ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology
Vol. 2, Issue 8, August 2015

Review on “Hadamard Matrices its construction and some interesting properties"

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Abstract:- In this paper we introduce Hadamard matrix, its definition, examples, construction of Hadamard matrices, its some properties and conclusion.

Keywords: Hadamard Matrix

INTRODUCTION

A Hadamard matrix, Named after the French mathematician Jacques Hadamard, is square matrix whose entries are eithers +1 or -1 and whose rows are mutually orthogonal.

For Example $H_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \end{matrix}$

Clearly $R_1 \, R_2 = O$

So Rows are orthogonal

It is a Hadamard matrix of order 2 Columns in Hadamard matrix are also mutually orthogonal.

CONSTRUCTION

In 1867, James, Joseph Sylvester constructed Hadamard matrix in the following manner.

Replace 1 by H_2

And -1 by $-H_2 \; in \; H_2, we \; get \; H_4$

So $H_4 = \begin{bmatrix} H_2 & -H_2 \\ H_2 & H_2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} 4 \times 4$$

Clearly $H_4 \; is \; a \; Hadamard \; matrix \; of \; order \; 4. \; By \; using \; same \; process \; of \; replacing \; 1 \; by H_2$ and -1 by -H_2 in H_4 , We get H_8 , we get H_8 again a Hadamard matrix of order 8

It become a chain of Hadamard matrices very amazing

So $H_2 \rightarrow H_4 \rightarrow H_8 \rightarrow H_{16} \rightarrow$

↓ ↓
↓ ↓

Order 2 $4 = 2^2$ $8 = 2^3$ $16 = 2^4$

So Sylvester constructed Hadamard matrices of order 2^n where n is a tve integer properties

$$H_2^T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Then $H_2 H_2^T = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=2I_2$$

In general the property is

$$H_n H_n^T = nI_n$$

Where H_n is a Hadamard matrix.

Further if we multiply any row (column) of a Hadamard matrix -1 then the resulting matrix is again a Hadamard matrix.

If we interchange any two rows (or columns) of a Hadamard matrix, even then resulting matrix is a Hadamard matrix.

In 1893 Hadamard constructed matrices of order 12 and 20. So $H_{12}$and H_{20} are missing Hadamard matrices in the Sylvester constructed Hadamard matrices. $H_2, H_4, H_8, H_{16}, H_{32} \ldots$

Recently Hadi Kharaghani and Behruz Tayfeh Rezaie announced on 21st June 2004 that they constructed a Hadamard matrix of order 428. As of 2008 there are 13 multiples of 4 less than or equal to 2000 for which no Hadamard matrix of that order is known.

They are 668, 716, 892, 1004, 1132, 1244, 1388, 1436, 1676, 1772, 1916, 1948, 1964 .

Construction of a Hadamard matrix by J. Williamson method

First, we study some properties and some concepts required for the J. Williamson method.

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International Advanced Research Journal in Science, Engineering and Technology
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PROPERTY

If we multiply any row (column) of a Hadamard matrix by -1, then the resulting matrix is again a Hadamard matrix.

If we interchange any two row (or column) of a Hadamard matrix, even then the resulting matrix is a Hadamard matrix.

CIRCULANT MATRIX

The matrix of type $\begin{bmatrix} a & b & c \\ c & a & b \\ -b & c & a \end{bmatrix}$ is called a circulant matrix. It is denoted by circ (a b c).

Example $\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$ is a circulant matrix. It is also Hadamard matrix.

It is the only known circulant matrix, which is a Hadamard matrix.

Product (or sum) of two circulant matrices is a circulant matrix.

Inverse of a circulant matrix is a circulant matrix.

Let □□ = circ (0 1 0)

then □^{2} = □ · □

$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

= circ (0 0 1)

Also□ □^{3} = □$^□$ □

= circ (1 0 0)

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$

∴ □^{3} = I

⇒ □$^□$. □ = I

∴ □$^{□□}$ = □$^□$

Also □^{T} $= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

∴ □$^□$ = □$^□$

∴ From above, we have

□$^□$ = □$^{□□}$= □$^{□□}$ where □ = circ (0 1 0)

WILLIAMSON'S METHOD STATEMENT

If A, B, C and D are four symmetric and circulant matrices of order t whose entries are +1 and -1 such that it satisfies

$$A^2 + B^2 + C^2 + D^2 = 4t\ I_t$$

Then $H = \begin{pmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{pmatrix}$ because Hadamard matrix.

The matrices A, B, C and D are known as Williamson's matrices of order t. Where as H is a Hadamard matrix of order 4t.

To construct a Hadamard matrix of order 12.

Order = 12 = 4 (3)

∴ we take t = 3

Let □ = circ (0 1 0) then □^{2} = circ (0 0 1)

∴ □□□□□$^□$ = circ (0 1 0) + circ (0 0 1)

= circ (0 1 1)

$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

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Now neither $\square$ nor $\square^2$ are symmetric matrix.

but $\square\square\square\square\square^{\square\square}$ is a symmetric matrix.

Let $A = \square^3 + \square + \square^2$

or $A = I + \square + \square^2$

or $A = I + w_1$ where $w_1 = \square + \square^2$

Also Let $B = I - w_1$

$C = I - w_1$

$D = I - w_1$

Then clearly A, B, C and D are all symmetric as well as circulant matrices with entries +1 or -1.

Now $w_1^2 = (\square + \square^2)^2 = (\square + \square^{-1})^2 = \square^{\square} + \square^{-2} + 2\square\square^{-1}$

$= \square^{-1} + \square + 2I$

$= w_1 + 2I$

$\therefore$ $A^2 = (I + w_1)^2 = I + 2w_1 + w_1^2$

$= I + 2w_1 + w_1 + 2I$

$\therefore$ $A^2 = 3(I + w_1)$

Also $B^2 = I - 2w_1 + w_1^2$

$= I - 2w_1 + w_1 + 2I$

$\therefore$ $B^2 = 3I - w_1$

Also $C^2 = 3I - w_1$ & $D^2 = 3I - w_1$

Now $A^2 + B^2 + C^2 + D^2 = 12\,I$

$= 4\,(3)\,I$

$\therefore$ It satisfies $A^2 + B^2 + C^2 + D^2 = 4t\,I_t$

Now $A = I + w_1$

$$= \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} + \begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$$

and $B = I - w_1$ = circ (1 0 0) – cir (0 1 1)

= circ (1 -1 -1)

$$= \begin{bmatrix}1 & -1 & -1\\-1 & 1 & -1\\-1 & -1 & 1\end{bmatrix}$$

Similarly C = D = B = circ (1 -1 -1)

$\therefore$ Hadamard matrix of order 12 will be

$$\begin{pmatrix} A & B & C & D\\ -B & A & D & -C\\ -C & -D & A & B\\ -D & C & -B & A \end{pmatrix}$$

Put matrices A, B, C, D in above, we get H_{12}

If H is a Hadamard matrix of order n. Then

(i) $HH^T = nI_n$

(ii) $[detH] = n^{\frac{1}{2}n}$

(iii) $HH^T = H^T H$

(iv) Hadamard matrices may be changed into other Hadamard matrices by different arrangements of rows and column and by multiplying rows and column by -1. The matrices so obtained are known as H-equivalent.

(v) Every Hadmard martrix is H-equivalent to an Hadamard matrix which has every element of the its first row and column +1.

These latter matrices are called normalized.

(vi) If H is a normalized hadamard martrix of order 4n, them every row (column) except the first has 2n minus ones and 2n plus ones in each row (column). Further n minus ones in any row (column) overlap with n minus ones in each other row (column)

(vii) the order of an Hadamard matrix is 1, 2 or 4n, n positive integer.

Theorm If a Hadamard matrix of order n exists then n=1,2 or a multiple of 4.

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Suppose n>2 and standardize H_n

Permute columns so that

+ + . . . + +	+ + . . . + +	+ + . . . + +	+ + . . . + +
+ + . . . + +	+ + . . . + +	- - . . . - -	- - . . . - -
+ + . . . + +	- - . . . - -	+ + . . . + +	- - . . . - -
p	q	r	s

then p+q+r+s = n the length of the vectors

p+q – r – s = 0 Row1 and Row2 are orthogonal

p – q – r +s =0 Row 2 and Row3 are orthogonal

p – q + r – s = 0 Row 3 and Row 1 are orthogonal

Therefore, we have n = 4a

Also n = 4b = 4c= 4d

So if a Hadamard matrix of order n exists then the order n must be either 1,2 or a multiple of 4.

APPLICATIONS

Hadamard matrices have application in Error – correcting codes, Modern CDMA Cellphones, pattern recognition, neuroscience optical communication and information hiding.

CONCLUSION

Although Hadamard matrices look simple but have interesting properties and very productive applications.

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