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Fixed Point Results for Weak Compatible Mapping in Fuzzy 2-Metric Space

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Abstract: In this paper we have proved some fixed point result in complete fuzzy 2-metric space for weak compatible mappings.

Keywords: Fixed Point, Fuzzy Metric Space, Fuzzy 2-Metric Space, Common Fixed Point, Weakly Compatible Mapping.
Subject classification: 47H10, 54H25.

I. INTRODUCTION

Zadeh [18] introduced the concept of fuzzy sets almost 50 years back in 1965, followed by many researchers [9, 10, 15, 16] they have studied fixed point theory in fuzzy metric spaces. The concept of fuzzy metric spaces also introduced ways by Erceg [4], Kaleva and Seikkala [12], Kramosil and Michalek [13] and Deng [3].
Earlier fuzzy mappings was studied by [1, 2, 11, 17] which opened a new vindo for further study and development of in analysis in such spaces and mappings with a vast applications. As a consequence many metric fixed point results were generalized to fuzzy metric spaces by various authors. Gahler in a series of papers [6, 7, and 8] investigated 2-metric spaces. Sharma, Sharma and Iseki [14] studied for the first time contraction type mappings in 2-metic space.
We know that 2-metric space is a real valued function of a point triples on a set X, which abstract properties were suggested by the area function in Euclidean spaces. In the present paper we obtain some common fixed point theorems on fuzzy metric spaces generalizing the earlier results of fisher [5], also we extend this result to fuzzy 2-metric spaces.

II. PRELIMINARIES

To start the main result we need some basic definitions.

Definition 2.1: A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian Topological monodies with unit 1 such that $a * b \geq c * d$ whenever $a \geq c$ and $b \geq d$ for all a, b, c, d $\in [0, 1]$
Example of t-norm are $a * b = a b$ and $a * b = \min\{a, b\}$

Definition 2.2: The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set,* is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions for all x, y, z $\in$ X and s, t > 0,

$$(FM-1): M(x,y,0)=0$$
$$(FM-2): M(x,y,t)=1, \forall t>0, \Leftrightarrow x=y$$
$$(FM-3): M(x,y,t)=M(y,x,t)$$
$$(FM-4): M(x,z,t+s)\geq M(x,y,t)*M(z,y,s)$$
$$(FM-5): M(x,y,a):[0,1)\to[0,1] \text{ is left continuous}$$

In what follows (X, M,*) will denote a fuzzy metric space. Note that M (x, y, t) can be thought of as the degree of nearness between x and y with respect to t. We identify x = y with M (x, y, t) = 1 for all t > 0 and M (x, y, t) = 0 with ∞.

Example 2.1: Let (X, d) be a metric space.
Define $a * b = a b$, or $a * b = \min\{a, b\}$ and for all x, y$\in$ X and t > 0,

$$M(x,y,t)=\frac{t}{t+d(x,y)} \qquad \text{---- 2.1.1}$$

Then (X, M,*) is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Definition 2.3: Let (X, M, *) is a fuzzy metric space.
(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, $\lim_{n\to\infty} M(x_n,x,t)=1$
(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p},x_n,t)=1, \forall t>0 \text{ and } p>0$$

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be Complete.
Let (X, M, *) is a fuzzy metric space with the following condition.

(FM-6) $\lim_{t\to\infty} M(x,y,t)=1, \forall x,y\in X$

Definition 2.4: A function M is continuous in fuzzy metric space iff whenever

$$x_n\to x, y_n\to y \Rightarrow \lim_{n\to\infty} M(x_n,y_n,t)\to M(x,y,t)$$

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Definition 2.5: Two mappings A and S on fuzzy metric space X are weakly commuting if and only if M (ASu, SAu, t) ≥ M (Au,Su,t) u∈X

Definition 2.6: A binary operation * : [0, 1] x [0,1] x [0,1] → [0,1] is called a continuous t-norm if ([0,1],*) is an abelian topological monodies with unit 1 such that $a_1 * b_1 * c_1 \geq a_2 * b_2 * c_2$ whenever $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ for all a_1, a_2, b_1, b_2 and c_1, c_2 are in [0,1].

Definition 2.7: The 3-tuple (X, M, *) is called a fuzzy 2-metric space if X is an arbitrary set, * is continuous t-norm and M is fuzzy set in X^3 x [0,∞) satisfying the following conditions:

$(FM' - 1): M(x, y, z, 0) = 0$

$(FM' - 2): M(x, y, z, t) = 1, \forall\ t > 0, \Leftrightarrow\ x = y$

$(FM' - 3): M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$

symmetry about three varriable

$(FM' - 4): M(x, y, z, t_{1,} t_{2,}, t_{3,}) \geq$

$M(x, y, u, t_{1,}) * M(x, u, z, t_{2,}) * M(u, y, z, t_{3,})$

$(FM' - 5): M(x, y, z): [0,1) \rightarrow [0,1]$

is left continuous $\forall x, y, z, u \in X,\ t_1, t_2, t_3 > 0$

Definition 2.8: Let (X, M,*) be a fuzzy 2-metric space:

(1) A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be convergent to a point x ∈ X,

$\lim_{n\to\infty} M(x_n, x, a, t) = 1,$ for all $a \in X$ and $t > 0$

(2) A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence, if

$\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1,$ for all $a \in X$ and $t, p > 0$

(3) A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.9: A function M is continuous in fuzzy 2-metric space, iff whenever

$x_n \rightarrow x,\ y_n \rightarrow y,$ then $\lim_{n\to\infty} M(x_n, y_n, a, t) = M(x, y, a, t),$

$\forall\ a \in X$ and $t > 0$

Definition 2.10: Two mappings A and S on fuzzy 2-metric space X are weakly commuting iff M (ASu, SAu,a, t) ≥ M (Au,Su,a,t).

Some Basic Results

Lemma (2.1): For all x, y ∈ X, M(x, y) is non -decreasing.

Lemma (2.2): Let $\{y_n\}$ be a sequence in a fuzzy metric space (X, M,*) with the condition **(FM -6**) If there exists a number q ∈ (0,1) such that

$M(y_{n+2}, y_{n+1}, qt) \geq M(y_{n+1}, y_n, t)$, $\forall t > 0$ and

$n = 1, 2, 3......$, then $\{y_n\}$ is a Cauchy sequence in X.

Lemma (2.3): If for all x, y ∈ X, t > 0 and for a number q ∈ (0, 1),

$M(x, y, qt) \geq M(xc, y, t)$, then $x = y$

Fisher [5] proved the following theorem for three mappings in complete metric space:

Theorem 2.A: Let S and T be continuous mappings of a complete metric space (X, d) into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into S (X) ∩ T (X), which commutes with S and T and satisfy

d (Ax, Ay) ≥ α d (Sx, Ty) for all x, y ∈ X and 0 < α < 1.

Then S, T and A have a unique common fixed point.

III. MAIN RESULTS

Now we prove these theorems in complete fuzzy 2-metric space.

Theorem 3.1: Let (X, M, *) be a complete fuzzy 2-metric space and let S^r and T^r be continuous mappings of X in X, then S^r and T^r have a common fixed point in X if there exists continuous mapping A^r of X into S^r (X) ∩ T^r (X) which weakly compatible with S^r and T^r and

3.1.1.

$$M(A^r x, A^r y, q^r t, a) \geq \left\{ \begin{matrix} M(T^r y, A^r y, t, a) * M(S^r x, A^r x, t, a) * \\ M(S^r x, T^r y, a, t) * M(A^r x, T^r y, a, t) * M(S^r x, A^r y, a, t) \end{matrix} \right\}$$

for all x, y, a ∈ X, t > 0, and 0 < q < 1. And

3.1.2. $\lim_{n\to\infty} M(x, y, z, a, t) = 1, \forall x, y, z, a \in X$

Then S^r, T^r and A^r have a unique common fixed point.

Proof: We define a sequence $\{x_n\}$ such that $A^r x_{2n} = S^r x_{2n-1}$ and $A^r x_{2n-1} = T^r x_{2n}$, n = 1, 2,-----

We shall prove that $\{A^r x_n\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1.1), we write

$$M(A^r x_{2n}, A^r x_{2n+1}, a, q^r t) \geq \left\{ \begin{matrix} M(T^r x_{2n+1}, A^r x_{2n+1}, a, t) * M(S^r x_{2n}, A^r x_{2n}, a, t) * \\ M(S^r x_{2n}, T^r x_{2n+1}, a, t) * M(A^r x_{2n}, T^r x_{2n+1}, a, t) * \\ M(S^r x_{2n}, A^r x_{2n+1}, a, t) \end{matrix} \right\}$$

$$M(A^r x_{2n}, A^r x_{2n+1}, a, q^r t) \geq \left\{ \begin{matrix} M(A^r x_{2n}, A^r x_{2n+1}, A^r, t) * M(A^r x_{2n+1}, A^r x_{2n}, a, t) * \\ M(A^r x_{2n+1}, A^r x_{2n}, a, t) * M(A^r x_{2n}, A^r x_{2n}, a, t) * \\ M(A^r x_{2n+1}, A^r x_{2n+1}, a, t) \end{matrix} \right\}$$

$$\Rightarrow M(A^r x_{2n}, A^r x_{2n+1}, a, q^r t) \geq M(A^r x_{2n-1}, A^r x_{2n}, a, q^r t)$$

$$= \left\{ \begin{matrix} M(A^r x_{2n}, A^r x_{2n+1}, a, t) * M(A^r x_{2n+1}, A^r x_{2n}, a, t) * \\ M(A^r x_{2n+1}, A^r x_{2n}, a, t) * 1 * 1 \end{matrix} \right\}$$

$$\geq \left\{ \begin{matrix} M(A^r x_{2n-1}, A^r x_{2n}, a, \frac{t}{q^r}) * M(A^r x_{2n}, A^r x_{2n-1}, a, \frac{t}{q^r}) * \\ M(A^r x_{2n}, A^r x_{2n-1}, a, \frac{t}{q^r}) * 1 * 1 \end{matrix} \right\}$$

Therefore

$$M(A^r x_{2n}, A^r x_{2n+1}, a, q^r t) \geq M\left(A^r x_{2n-1}, A^r x_{2n}, a, \frac{t}{q^r}\right)$$

By induction

$$M(A^r x_{2k}, A^r x_{2m+1}, a, q^r t) \geq M\left(A^r x_{2m}, A^r x_{2k-1}, a, \frac{t}{q^r}\right)$$

For every k and m in N, Further if 2m + 1 > 2k, then

$$M\left(A^r x_{2k}, A^r x_{2m+1}, a, q^r t\right) \geq M\left(A^r x_{2k-1}, A^r x_{2m}, a, \frac{t}{q^r}\right) \geq \ldots\ldots$$

$$\ldots\ldots\ldots\ldots \geq M\left(A^r x_0, A^r x_{2m+1-2k}, a, \frac{t}{q^{2kr}}\right) \quad \cdots\cdots(3.1.3)$$

If 2k > 2m+1, then

$$M\left(A^r x_{2k}, A^r x_{2m+1}, a, q^r t\right) \geq M\left(A^r x_{2k-1}, A^r x_{2m}, a, \frac{t}{q^r}\right) \geq \ldots\ldots$$

$$\geq M\left(A^r x_{2k-(2m+1)}, A^r x_0, a, \frac{t}{q^{(2m+1)r}}\right) \cdots\cdots(3.1.4)$$

By simple induction with (3.1.3) and (3.1.4) we have

$$M\left(A^r x_n, A^r x_{n+p}, q^r t\right) \geq M\left(A^r x_0, A^r x_p, \frac{t}{q^{nr}}\right).$$

For n = 2k, p = 2m+1 or n = 2k+1, p = 2m +1 and by (FM-4)

$$M\left(A^r x_n, A^r x_{n+p}, a, q^r t\right) \geq \left\{M\left(A^r x_0, A^r x_1, a, \frac{t}{2q^{nr}}\right) * M\left(A^r x_1, A^r x_p, a, \frac{t}{q^{nr}}\right)\right\} \ldots\ldots(3.1.5)$$

If n = 2k, p = 2m or n = 2k+1, p = 2m

Therefore every positive integer p and n in N

$$M\left(A^r x_0, A^r x_p, a, \frac{t}{q^{nr}}\right) \to 1 \quad as \quad n \to \infty$$

Thus $\{A^r x_n\}$ is a Cauchy sequence. Since the space X is complete there exists z ∈ X, such that

$$\lim_{n\to\infty} A^r x_n = \lim_{n\to\infty} S^r x_{2n-1} = \lim_{n\to\infty} T^r x_{2n} = z$$

It follows that $A^r z = S^r z = T^r z$ and therefore

$$M\left(A^r z, A^{2r} z, a, q^r t\right) \geq \begin{Bmatrix} M\left(T^r A^r z, A^r A^r z, a, t\right) * M\left(S^r z, A^r z, a, t\right) * M\left(S^r z, T^r A^r z, a, t\right) * \\ M\left(A^r z, T^r A^r z, a, t\right) * M\left(S^r z, A^r A^r z, a, t\right) \end{Bmatrix}$$

$$M(A^r z, A^{2r} z, a, q^r t) \geq M(S^r z, T^r A^r z, a, t)$$

$$\geq M(S^r z, A^r T^r z, a, t) \geq M(A^r z, A^{2r} z, a, t) ---- \geq M(A^r z, A^{2r} z, a, \frac{t}{q^{nr}})$$

$$\text{Since } \lim_{n\to\infty} M(A^r z, A^{2r} z, a \frac{t}{q^{nr}}) = 1 \Rightarrow A^r z = A^{2r} z$$

Thus z is common fixed point of A^r, S^r and T^r.

For **uniqueness**, let w (w ≠ z) be another common fixed point of S^r, T^r and A^r for all r > 0. By inequality we write

$$M\left(A^r z, A^r w, a, q^r t\right) \geq \begin{Bmatrix} M\left(T^r w, A^r w, a, t\right) * M\left(S^r z, A^r z, a, t\right) * M\left(S^r z, T^r w, a, t\right) * \\ M\left(A^r z, T^r w, a, t\right) * M\left(S^r z, A^r w, a, t\right) \end{Bmatrix}$$

$$M\left(A^r z, A^r w, a, q^r t\right) \geq \begin{Bmatrix} M\left(w, w, a, t\right) * M\left(z, z, a, t\right) * M\left(z, w, a, t\right) * \\ M\left(z, w, a, t\right) * M\left(z, w, a, t\right) \end{Bmatrix}$$

$$M\left(A^r z, A^r w, a, q^r t\right) \geq \left\{M\left(z, w, a, t\right)\right\}$$

$$\Rightarrow M\left(z, w, a, q^r t\right) \geq \left\{M\left(z, w, a, t\right)\right\}$$

Therefore by lemma (2.3), we get z = w.

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