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Review of "Some operators on Hilbert Space and their Spectrum"

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Abstract: In the paper there will be a brief introduction of Hilbert space then some types of operators on Hilbert space and their spectrum.

Keywords: Hilbert space, operators, spectrum

INTRODUCTION

A Hilbert space is a vector space H together with an inner product <x, y> defined on it such that the norm IIxII= $\sqrt{<x,x>}$ defined on H makes it a complete metric space. So a Hilbert space (H, < >) is a complete inner product space.

An operator on a Hilbert space H is a continuous or bounded linear transformation T:H→ H. The set of all operators on H is denoted by B (H).

Spectrum of an operator T:H→ H is the set of all eigen values of T and is denoted by σ(T) = { $\lambda : T-\lambda I$ is singular}

MATHEMATICAL DISCUSSION

Some types of operators are adjoint operator, self-adjoint, operator positive operator, Normal operator, Unitary operator.

ADJOINT OPERATOR

Let T be an operator on a Hilbert space H then there exists a unique operator denoted by T*: H→ H such that

<Tx, y> = <x, T*y> $\forall$ x, y $\in$ H.

The operator T* is called the adjoint of T.

These are some important properties of operators

(1) $(T_1 + T_2)^* = T_1^* + T_2^*$

(2) $(\alpha T)^* = \bar{\alpha} T^*$

(3) $(T_1 T_2)^* = T_2^* T_1^*$

(4) $T^{**} = T$

(5) IITII = IIT^*II

(6) <Tx, x> = 0 $\forall$ x $\in$ H $\Leftrightarrow$ T = 0

SELF-ADJOINT OPERATOR

An operator T on a Hilbert space H is said to be self-adjoint if T= T^*

So < Tx, y> = <x, Ty> $\forall$ x, y $\in$ H

Theorem. An operator T on a Hilbert space H is self-adjoint iff <Tx, x> is real $\forall$, x $\in$ H

Let T be self-adjoint operator on H.

$\therefore$ T* = T

Now < Tx, x> = <x, T^*x>

= <x, Tx>

<Tx, x> = $<\overline{Tx, x}>$

$\Rightarrow$ <Tx, x> is real $\forall$ x $\in$ H

conversely, suppose < Tx, x> $\in$ R $\forall$ x $\in$ H

Where R is the set of real numbers

$\therefore$ <Tx, x> = $<\overline{Tx, x}>$

= $<\overline{x, T^* x}>$

= <T*x, x> $\forall$ x $\in$ H

$\Rightarrow$ <Tx - T^*x, x> = 0 $\forall$ x $\in$ H

< (T-T*) x, x>=0 $\forall$ x $\in$ H

$\Rightarrow$ T-T^* = O

$\Rightarrow$ T=T^*

T is self-adjoint

Def If T_1 and T_2 are self-adjoint operator on a Hilbert space H

Define a relation $\leq$ on B(H) as $T_1 \leq T_2 \Leftrightarrow <T_1x, x> \leq <T_2x, x>$ $\forall$ x $\in$ H

POSITIVE OPERATOR

A self-adjoint operator T on a Hilbert space is said to be positive.

if T $\geq$0

i.e. <Tx, x> $\geq$ 0 $\forall$, x $\in$ H

NORMAL OPERATOR

An operator N on a Hilbert space H is said to be normal operator if $NN^* = N^*N$.

Theorem:- T is normal operator on H iff

$\|Tx\| = \|T^*x\| \ \forall\ x \in H$

T is normal

$\Leftrightarrow \quad TT^* = T^*T$

$\Leftrightarrow \quad TT^* - T^*T = 0$

$\Leftrightarrow \quad \langle (TT^* - T^*T)x, x\rangle = 0 \ \forall\ x \in H$

$\Leftrightarrow \quad \langle (TT^*x- x\rangle - \langle T^*Tx, x\rangle = 0$

$\Leftrightarrow \quad \langle (T^*x, T^*x \rangle - \langle Tx, Tx\rangle = 0$

$\Leftrightarrow \quad \langle (T^*x, T^*x\rangle = \langle Tx, Tx\rangle$

$\Leftrightarrow \quad \|T^*x\|^2 = \|Tx\|^2$

$\Leftrightarrow \quad \|T^*x\| = \|Tx\|$

UNITARY OPERATOR

An operator U on a Hilbert Space H is said to be unitary if $UU^* = U^*U = I$.

Theorem If T is a self-adjoint operator on H, then every eigen-value of T is real.

Let λ be an eigen value of T.

Then $\exists\ x \neq 0$ Such that $Tx = \lambda x$

Now $\langle Tx, x\rangle = \langle \lambda x, x\rangle$

$= \lambda \langle x,x\rangle$

$= \lambda \|x\|^2$

As T is self-adjoint

So $\langle Tx, x\rangle$ is real $\forall\ x \in H$

$\therefore\ \lambda$ is real.

So all eigen values of a self-adjoint operator T are real.

So spectrum of T i.e. $\sigma(T)$ is a subset of real numbers.

Theorem If T is a positive operator then every eigen value of T is positive.

As T is a positive operator

$\therefore\ \langle Tx, x\rangle \geq 0\ \forall\ x \in H$

As proved above $\langle Tx, x\rangle = \lambda \|x\|^2$

So $\lambda \geq 0$

Therefore each eigen value of a positive operator is positive.

so spectrum of a positive operator consists of positive real values

Theorem, If T is a unitary operator on a Hilbert space H then every eigen value of T has absolute value 1.

If T is unitary operator on a Hilbert space H then

$T^*T = I$

Now $\langle Tx, Ty\rangle = \langle T^*Tx, y\rangle$

$= \langle Ix, y\rangle$

$= \langle x, y\rangle$

$\Rightarrow \quad \langle Tx, Ty\rangle = \langle x, y\rangle \ \forall\ x, y \in H$

By taking y = x,

$\langle Tx, Tx\rangle = \langle x, x\rangle$

$\|Tx\|^2 = \|x\|^2$

$\Rightarrow \|Tx\|=\|x\|$

Let λ be an eigen value of T

i.e. $\lambda \in \sigma(T)$

Then $\exists\ x \neq 0$ such that

$Tx = \lambda x$

$\|\lambda\| = \|Tx\|$

$= \|x\|$

$\Rightarrow |\lambda| = 1$

So if $\lambda \in \sigma$ (T) and T is a unitary operator then absolute value of λ is 1.

Theorem If T is a self-adjoint or unitary operator on H then the eigen vectors of T, corresponding to distinct eigen values of T are orthogonal.

Let T be a self-adjoint or unitary operator on a Hilbert space H.

Let $\lambda_1 \in \sigma(T)$ and $\lambda_2 \in \sigma(T)$

$\therefore\ \exists\ x_1 \neq 0$

and $x_2 \neq 0$ such that

$Tx_1 = \lambda_1 x_1$

and $Tx_2 = \lambda_2 x_2$

Now if T is self-adjoint then both λ_1 and λ_2 are real.

So $\lambda_1 \langle x_1, x_2\rangle = \langle \lambda_1 x_1, x_2\rangle$

$= \langle Tx_1, x_2\rangle$

$= \langle x_1 , T^* x_2\rangle$

$= \langle x_1 , T x_2\rangle$

$= \langle x_1 , \lambda_2 x_2\rangle$

$= \overline{\lambda_2} \langle x_1 , x_2\rangle$

$= \lambda_2 \langle x_1 ,x_2\rangle$

$(\lambda_1 - \lambda_2) \langle x_1, x_2\rangle = 0$

As $\lambda_1 \neq \lambda_2$

$\therefore\ \langle x_1, x_2\rangle = 0$

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So eigen vectors corresponding to distinct eigen values are orthogonal.

Further on assuming T as unitary operator.

We have T*T = I

$\therefore \langle x_1, x_2\rangle = \langle x_1, T^*Tx_2\rangle$

$= \langle Tx_1, Tx_2\rangle$

$= \langle \lambda_1 x_1, \lambda_2 x_2\rangle$

$= \lambda_1 \overline{\lambda_2} \langle x_1, x_2\rangle$

$(1 - \lambda_1 \overline{\lambda_2}) \langle x_1, x_2\rangle = 0$

Now $\lambda_2 \overline{\lambda_2} = I \lambda_2 I^2$

= 1 as T is unitary

So $\overline{\lambda_2} = \dfrac{1}{\lambda_2}$

$\therefore \lambda_1 \overline{\lambda_2} = \dfrac{\lambda_1}{\lambda_2}$

$\neq 1$ as $\lambda_1 \neq \lambda_2$

So $\langle x_1, x_2\rangle = 0$

Again eigen vectors of a unitary operator T are orthogonal corresponding to distinct eigen values of T.

Theorem If T is a normal operator on H then x is an eigen vector of T with eigen value λ iff x is an eigen vector of T^* with eigen value $\overline{\lambda}$

As T is a normal operator

$\therefore$ TT* = T* T

Also for any scalar λ

$(T-\lambda I)^* = T^* - (\lambda I)^*$

$= T^* - \overline{\lambda} I^*$

$= T^* - \overline{\lambda} I$

Now $(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)(T^* - \overline{\lambda} I)$

$= TT^* - \lambda T - \lambda T^* + \lambda \overline{\lambda}$

$= TT^* - \lambda T - \lambda T^* + I \lambda I^2$

Also $(T-\lambda I)^* (T-\lambda I) = (T^* - \overline{\lambda} I)(T-\lambda I)$

$= T^*T - \lambda T^* - \overline{\lambda} T - \lambda \overline{\lambda}$

$= T^*T - \lambda T^* - \overline{\lambda} T + I \lambda I^2$

As T is normal

$= TT^* - \lambda T^* - \overline{\lambda} T + I \lambda I^2$

So $(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)^*(T-\lambda I)$

$\therefore$ $T-\lambda I$ is a normal operator for any scalar λ

Now $(T-\lambda I)^* = T^* - \overline{\lambda} I^*$

$= T^* - \overline{\lambda} I$

As $T-\lambda I$ is normal

$\therefore \| (T-\lambda I)x \| = \| (T-\lambda I)^* x \| \ \forall\ x \in H$

$\Leftrightarrow \| (T-\lambda I)x \| = \| (T^* - \overline{\lambda} I)x \| \ \forall\ x \in H$

$\Leftrightarrow \| (Tx - \lambda x) \| = \| (T^*x - \overline{\lambda} x \|$

So if x is an eigen vector of normal operator T with eigen value λ then x is an eigen vector of adjoint operator T^* with eigen value $\overline{\lambda}$.

Theorem If T is a normal operator on a Hilbert space H then eigen vectors of T corresponding to distinct eigen values are orthogonal to each other.

Let T be normal operator

Let λ_1 and λ_2 belongs to $\sigma(T)$ and $\lambda_1 \neq \lambda_2$

So $\exists\, x_1 \neq 0$ and $x_2 \neq 0$

such that

$Tx_1 = \lambda_1 x_1$

and $Tx_2 = \lambda_2 x_2$

$\therefore \lambda_1 \langle x_1, x_2\rangle = \langle \lambda_1 x_1, x_2\rangle$

$= \langle Tx_1, x_2\rangle$

$= \langle x_1, T^*x_2\rangle$

$= \langle x_1, \overline{\lambda}_2 x_2\rangle$

$= \lambda_2 \langle x_1, x_2\rangle$

$(\lambda_1 - \lambda_2) \langle x_1, x_2\rangle = 0$

$\Rightarrow \langle x_1, x_2\rangle = 0$

So eigen vectors of a normal operator T are orthogonal corresponding to distinct eigen values.

CONCLUSION

After applying definitions and results, we come to conclusion that operators on Hilbert space show very interesting spectral values. Self-adjoint operators always have advantage of real eigen values. Positive operators have positive real eigen values. Unitary operators have eigen values having absolute value unity. Further eigen vectors corresponding to distinct eigen values of self-adjoint, unitary and normal operators are pairwise orthogonal.

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