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Correspondence and Isomorphism Theorems for Generalized Soft Groups

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Abstract: The aim of this paper is to generalize the existing correspondence and isomorphism theorems of groups in the crisp setup to those of generalized soft group.

Keywords: (Generalized) Soft set, (Generalized) Soft map, (Generalized) Soft group, (Generalized) Soft homomorphism, (Generalized) Soft (pure) isomorphism

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I. INTRODUCTION

Ever since Molodtsov [7] introduced the notion of a soft set over a universal set as a parametrized family of subsets of a given universal set to model some types of uncertainties, mathematicians started imposing and studying algebraic, topological and topologically algebraic structures on them.

In fact, a soft (normal) subgroup over a group is a parametrized family of (normal) subgroups of the given group, a soft (left ideal, right ideal, ideal) subring over a ring is a parametrized family of (left ideals, right ideals, ideals) subrings of the given ring etc.. In general any (algebraic, topological or topologically algebraic) soft sub object over a given (algebraic, topological or topologically algebraic) object may be defined as a parametric family of the corresponding (algebraic, topological or topologically algebraic) sub objects of the given (algebraic, topological or topologically algebraic) object. Thus mathematically speaking, a soft subobject over a crisp object U is a pair (F, E) where E is a parameter set and $F: E \to P(U)$ is such that for each e in E, $F(e)$ is a crisp subobject of U.

In fact, the notion of soft object over a crisp object U is a beautiful natural generalization of the crisp sub object, as the collection of all *non-empty* soft sub objects over U with a given parameter set over a given crisp object is in one-one correspondence with the collection of all crisp sub objects of the given object if the parameter set is a singleton set. Notice that the empty soft object is the unique one with the parameter set, the empty set as, for a soft object (F, E) over an object U, E is empty implies F is empty.

Murthy-Gouthami[12] introduced the notions of generalized soft group, generalized soft (normal) subgroup, generalized soft quotient group etc., generalizing the corresponding notions of a soft group over a group and showed that several of the crisp group theoretic results naturally extended to these new objects too.

Further, in another paper Murthy-Gouthami[13] introduced the notions of generalized soft homomorphism of generalized soft groups, generalized soft (inverse) image of generalized soft (normal) subgroup under generalized soft homomorphism, generalized soft kernel of a generalized soft homomorphism etc., generalizing the corresponding existing notions for a soft group over a group and showed that several of the crisp theoretic results naturally extend to these new objects too.

Now our aim in this paper is to generalize the existing correspondence and isomorphism theorems in the crisp setup, to those of generalized soft group.

II. PRELIMINARIES

(A) **Groups:** (a) (i) For any map $f: X \to Y$, for all a, b in X, $a \sim b$ iff $fa = fb$ is an equivalence relation on X with eqivalence classes $(f^{-1}fa)_{a \in X}$, also called *kernel classes*. (ii) G be a group. For any group homomorphism $\phi: G \to H$, $Ker\phi \subseteq C$ and C is a subgroup of G imply $\phi^{-1}\phi C = C$ (iii) For any subgroup A of G, for any normal subgroup H of G, for any subgroup B of H such that B is a normal subgroup of A we have $\phi: \frac{A}{B} \to \frac{G}{H}$ defined by $\phi(Ba) = Ha$ for all $a \in B$ is a group homomorphism such that $\phi(\frac{A}{B}) = \frac{AH}{H}$. If $B = H$ then $\frac{A}{B} = \frac{A}{H}$ is a subgroup of $\frac{G}{H}$. Further, if A is normal in G then $\frac{A}{H}$ is normal in $\frac{G}{H}$. (iv) for any pair of subgroups H, K of G such that H is a normal subgroup of K and A is a normal

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subgroup of G, we have $\frac{HA}{A}$ is a normal subgroup of $\frac{KA}{A}$ (v) for any subgroup K of G and for any pair of normal subgroups H, A of K, we have HA is a normal subgroup of KA. (vi) for any (normal) subgroup H of G and for any normal subgroup K of G such that $K \subseteq H$, we have $\frac{H}{K}$ is a (normal) subgroup of $\frac{G}{K}$.
(b) Generalized first isomorphism theorem: For any group homomorphism $\phi: G \to H$ & for any subgroup A of G we have (1) $\phi|A: A \to \phi A$ is an epimorphism (2) $Ker(\phi|A) = Ker(\phi) \cap A$ (3) $\frac{A}{Ker(\phi|A)}$ which is isomorphic to $(\phi|A)(A) = \phi A$.
(c) Generalized second isomorphism theorem: For any group G, and for any subgroup A of G, for any normal subgroup B of G, $F = F_{AB}: \frac{AB}{B} \to \frac{A}{A\cap B}$ defined by $abB = aB \to a(A \cap B) = F_{AB}a$ or Fa defines an isomorphism such that for all C is a subgroup of A, $F|\frac{CB}{B}: \frac{CB}{B} \to \frac{C(A\cap B)}{A\cap B}$ is a monomorphism such that $F(\frac{CB}{B}) = \frac{C(A\cap B)}{A\cap B}$ or $F|\frac{CB}{B}: \frac{CB}{B} \to \frac{C(A\cap B)}{A\cap B}$ is an isomorphism.
(d) Generalized third isomorphism theorem: For any group G and for any normal subgroups A, B of G such that A is a subgroup B and C is a subgroup of G, $\frac{CB}{B}$ is isomorphic to $\frac{\frac{CA}{A}\frac{B}{A}}{\frac{B}{A}}$.

(B) **Soft Sets** In what follows we recall the following basic definitions from the Soft Set Theory which are used in the main results: (e) [7] Let U be a universal set, $P(U)$ be the power set of U and E be a set of parameters. A pair (F, E) is called a *soft set* over U iff $F: E \to P(U)$ is a mapping defined by for each $e \in E$, $F(e)$ is a subset of U. In other words, a soft set over U is a parametrized family of subsets of U.
Notice that a collective presentation of all the notions, soft sets and gs-sets raised some serious notational conflicts and to fix the same Murthy-Maheswari[6] deviated from the above notation for a soft set and adapted the following notation for convenience as follows:

Let U be a universal set. A typical *soft set* over U is an ordered pair (San Serif) $S = (\sigma_S, S)$, where S is a set of *parameters*, called the *underlying parameter set* for S, $P(U)$ is the power set of U and $\sigma_S: S \to P(U)$ is a map, called the *underlying set valued map* for S. Some times σ_S is also called the *soft structure* on S.

(f) [10] The *empty* soft set over U is a soft set with the empty parameter set, denoted by $\Phi = (\sigma_\phi, \phi)$. Clearly, it is unique.
(g) [9] A soft set S over U is said to be a *whole* soft set, denoted by U_S, iff $\sigma_S s = U$ for all $s \in S$. (h) [10] A soft set S over U is said to be a *null* soft set, denoted by Φ_S, iff $\sigma_S s = \phi$, the empty set, for all $s \in S$. Notice that $\Phi_\phi = \Phi$, the empty soft (sub) set.
For any pair of soft sets A, B over U,
(i) [8] A is a *soft subset* of B, denoted by $A \subseteq B$, iff (i) $A \subseteq B$ (ii) $\sigma_A a \subseteq \sigma_B a$ for all $a \in A$. The set of *all soft subsets* of B is denoted by $\mathcal{S}_U(B)$
(j) The following are easy to see:
(i) Always the empty soft set Φ is a soft subset of every soft set A
(ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$ iff $A = B$ and $\sigma_A a = \sigma_B a$ for all $a \in A$.
(k) For any family of soft subsets $(A_i)_{i\in I}$ of S,
(i) the *soft union* of $(A_i)_{i\in I}$, denoted by $\cup_{i\in I} A_i$, is defined by the soft set A, where (i) $A = \cup_{i\in I} A_i$ (ii) $\sigma_A a = \cup_{i\in I_a} \sigma_{A_i} a$, where $I_a = \{i \in I / a \in A_i\}$, for all $a \in A$
(ii) the *soft intersection* of $(A_i)_{i\in I}$, denoted by $\cap_{i\in I} A_i$, is defined by the soft set A, where (i) $A = \cap_{i\in I} A_i$ (ii) $\sigma_A a = \cap_{i\in I} \sigma_{A_i} a$ for all $a \in A$.
Notice that $\cap_{i\in I} A_i$ can become empty soft set.

(C) **Soft Groups, Soft Group homomorphisms:** In this section we first recall the existing notions of a soft group, soft (normal) subgroup, soft group homomorphism etc.. According to Aktas-Cagman[2], (1) if (F, A) is a soft set over a group G, then (F, A) is said to be a *soft group* over G if and only if $F(x) \leq G$ for all $x \in A$. (2) Let (F, A) and (H, K) be two soft groups over G. Then (H, K) is a *soft subgroup* of (F, A), written as $(H, K) \leq (F, A)$, if $K \subseteq A$, $H(x) \leq F(x)$ for all $x \in K$. (3) Let (F, A) be a soft group over G and (H, B) be a soft subgroup of (F, A). Then we say that (H, B) is a *normal soft subgroup* of (F, A), written $(H, B) \trianglelefteq (F, A)$, if $H(x)$ is a normal subgroup of $F(x)$ i.e., $H(x) \trianglelefteq F(x)$, for all $x \in B$.
(4) According to Sezgin-Atagun[1], if G is a group and (F, A) is a non-null soft set over G, then (F, A) is called a *normalistic soft group* over G if $F(x)$ is a normal subgroup of G for all $x \in Supp(F, A)$.
(5) According to [2], let (F, A) and (G, B) be two groups over G and K respectively, and let $f: G \to K$ and $g: A \to B$ be two functions. Then we say that (f, g) is a *soft homomorphism*, and that (F, A) is soft homomorphic to (H, B). The latter is written as $(F, A) \sim (H, B)$, if the following are satisfied: f is a homomorphism from G onto K, g is a mapping from A onto B, and $f(F(x)) = H(g(x))$ for all $x \in A$. In this definition, if f is an isomorphism from G to K and g is a one-to-

one mapping from A onto B. then we say that (f, g) is a *soft isomorphism* and that (F, A) is soft isomorphic to (G, B). The latter is denoted by $(F, A) \simeq (H, B)$.

(D) **Generalized soft sets**

In this section we recall the notions of generalized soft set or gs-set or s-set for short, gs-subset or s-subset, gs-union or s-union, gs-intersection or s-intersection etcetera from Murthy-Maheswari[5] and (increasing, decreasing, preserving) s-map, (inverse) image of s-subset under an s-map from Murthy-Gouthami[13].

From now on, the script letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ etc. denote s-sets and/or their subsets and any such script letter $\mathcal{Q}$ stands for the triplet $(Q, \overline{Q}, P(U_Q))$. A power algebra is boolean algebra of the form $P(U_A)$ for some set U_A.

(l) A *generalized soft set* or an *s-set* in short, is any triplet $\mathcal{A}$, where A is the *underlying set of parameters for* U_A *or parameter set* in short, $P(U_A)$ is *the complete lattice of all subsets of* U_A *parametrized under* $\overline{A}$ *with parameters from* A and $\overline{A}: A \to P(U_A)$ is the *underlying parametrizing map for* U_A.

(m) The s-set $\mathcal{A}$, where A = ∅, the empty set with *no* elements,
$P(U_A) = \{∅\}$, and $\overline{A}$ = ∅, the empty map, is called the *empty s-set* and is denoted by ∅.

(n) An s-set $\mathcal{A}$ is said to be a *whole s-set* iff the parametrizing map $\overline{A}: A \to P(U_A)$ is defined by $\overline{A}a = U_A$ for all $a \in A$.

(o) An s-set $\mathcal{A}$ is said to be a *null s-set* iff the parametrizing map $\overline{A}: A \to P(U_A)$ is defined by $\overline{A}a$ = ∅, the empty set, for all $a \in A$.

For any pair of s-sets $\mathcal{A}$ and $\mathcal{B}$,

$\mathcal{A}$ is an *s-subset* of $\mathcal{B}$, denoted by $\mathcal{A} \subseteq \mathcal{B}$, iff (i) $A \subseteq B$ (ii) $U_A \subseteq U_B$ or equivalently $P(U_A)$ is a complete ideal of $P(U_B)$ and (iii) $\overline{A}a \subseteq \overline{B}a$ for all $a \in A$.

The set of all s-subsets of the s-set $\mathcal{B}$ is denoted by $\mathcal{S}(\mathcal{B})$.

An s-subset $\mathcal{S}$ is *degenerated* iff S = ∅ and $\overline{S}$ = ∅, the empty map. Clearly, the empty s-set is degenerated. Note that degenerated s-subset is *not* unique.

(p) The following are easy to see:

(i) Always the empty s-set ∅ is an s-subset of every s-set $\mathcal{A}$.

(ii) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$ iff $A = B$, $U_A = U_B$ and $\overline{A} = \overline{B}$.

For any family of s-subsets $(\mathcal{A}_i)_{i \in I}$ of $\mathcal{B}$,

(q) the *s-union* of $(\mathcal{A}_i)_{i \in I}$, denoted by $\cup_{i \in I} \mathcal{A}_i$, is defined by the s-set $\mathcal{A}$, where

(i) $A = \cup_{i \in I} A_i$ is the usual set union of the collection $(A_i)_{i \in I}$ of subsets of B.

(ii) $P(U_A) = \vee_{i \in I} P(U_{A_i}) = P(\cup_{i \in I} U_{A_i})$, where $\vee_{i \in I} P(U_{A_i})$ is the complete ideal generated by $\cup_{i \in I} P(U_{A_i})$ in $P(U_A)$ which is the same as $P(\cup_{i \in I} U_{A_i})$

(iii) $\overline{A}: A \to P(U_A)$ is defined by $\overline{A}a = \cup_{i \in I_a} \overline{A}_i a$, where $I_a = \{i \in I | a \in A_i\}$

(r) the *s-intersection* of $(\mathcal{A}_i)_{i \in I}$, denoted by $\cap_{i \in I} (\mathcal{A}_i)$, is defined by the s-set $\mathcal{A}$, where

(i) $A = \cap_{i \in I} A_i$ is the usual set intersection of the collection $(A_i)_{i \in I}$ of subsets of B(ii) $P(U_A) = \cap_{i \in I} P(U_{A_i}) = P(\cap_{i \in I} U_{A_i})$ is the usual intersection of the complete ideals of $P(U_{A_i})_{i \in I}$ in $P(U_A)$

(iii) $\overline{A}: A \to P(U_A)$ is defined by $\overline{A}a = \cap_{i \in I} \overline{A}_i a$.

(s) For any pair of s-sets $\mathcal{A}$ and $\mathcal{B}$, an *s-map* is any pair (f, F), denoted by $\mathcal{F}$, where $f: A \to B$ and $F: P(U_A) \to P(U_B)$ is onto and extends $F|U_A$ or equivalently $F = P(F|U_A)$.

Note that quite often in all our examples $F|U_A$ will be denoted by F_0.

In what follows we give an Example to show that if F is *not* onto then some of the crucial properties like for subsets $\mathcal{C}, \mathcal{D}$ of $\mathcal{B}$, $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{B}$ implies $\mathcal{F}^{-1}\mathcal{C} \subseteq \mathcal{F}^{-1}\mathcal{D}$; $\mathcal{F}^{-1}(\mathcal{C} \cup \mathcal{D}) = \mathcal{F}^{-1}\mathcal{C} \cup \mathcal{F}^{-1}\mathcal{D}$; $\mathcal{F}^{-1}(\mathcal{C} \cap \mathcal{D}) = \mathcal{F}^{-1}\mathcal{C} \cap \mathcal{F}^{-1}\mathcal{D}$ etc. do *not* hold and, as we know, without them nothing much can be done:

Example 1: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be given by $\mathcal{A} = (\{p\}, \{(p, \{x, y, z\}), P(\{x, y, z\})\})$, $\mathcal{B} = (\{q\}, \{(q, \{a, b, c\}), P(\{a, b, c\})\})$, $f = \{(p, q)\}$ and $F = \{(∅, ∅), (x, a), (y, b), (z, b), (\{x, y\}, \{a, b\}), (\{z, x\}, \{a, b\}), (\{y, z\}, \{b\}), (\{x, y, z\}, \{a, b\})\}$. Then $F_0 = \{(x, a), (y, b), (z, b)\}$.

(1) Let $\mathcal{C} = (\{q\}, \{(q, \{b\}), P(\{a, b, c\})\})$, $\mathcal{D} = (\{q\}, \{(q, \{b, c\}), P(\{a, b, c\})\})$.

Let $\mathcal{F}^{-1}\mathcal{C} = \mathcal{M}$. Then $M = f^{-1}C = \{p\}$, $U_M = F_0^{-1}U_C = \{x, y, z\}$ and $\overline{M}p = \overline{A}p \cap\cup F^{-1}\overline{C}fp = \{x, y, z\} \cap \{y, z\} = \{y, z\}$.

Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{N}$. Then $N = f^{-1}D = \{p\}$, $U_N = F_0^{-1}U_D = \{x, y, z\}$ and $\overline{N}p = \overline{A}p \cap\cup F^{-1}\overline{C}fp = \{x, y, z\} \cap ∅ = ∅$, implying $\mathcal{C} \subseteq \mathcal{D}$ but $\mathcal{F}^{-1}\mathcal{C} \nsubseteq \mathcal{F}^{-1}\mathcal{D}$.

(2) Let $\mathcal{C} = (\{q\}, \{(q, \{b\})\}, P(\{a, b, c\}))$, $\mathcal{D} = (\{q\}, \{(q, \{c\})\}, P(\{a, b, c\}))$.

Then $\mathcal{C} \cup \mathcal{D} = (\{q\}, \{(q, \{b, c\})\}, P(\{a, b, c\}))$, $\mathcal{F}^{-1}\mathcal{C} = (\{p\}, \{(p, \{y, z\})\}, P(\{x, y, z\}))$, $\mathcal{F}^{-1}\mathcal{D} = (\{p\}, \{(p, ∅)\}, P(\{x, y, z\}))$, $\mathcal{F}^{-1}\mathcal{C} \cup \mathcal{F}^{-1}\mathcal{D} = (\{p\}, \{(p, \{y, z\})\}, P(\{x, y, z\}))$ and $\mathcal{F}^{-1}(\mathcal{C} \cup \mathcal{D}) = (\{p\}, \{(p, ∅)\}, P(\{x, y, z\}))$, implying $\mathcal{F}^{-1}(\mathcal{C} \cup \mathcal{D}) \neq \mathcal{F}^{-1}\mathcal{C} \cup \mathcal{F}^{-1}\mathcal{D}$.

(3) Let $\mathcal{C} = (\{q\}, \{(q, \{b, c\})\}, P(\{a, b, c\}))$, $\mathcal{D} = (\{q\}, \{(q, \{a, b\})\}, P(\{a, b, c\}))$.

Then $\mathcal{C} \cap \mathcal{D}$ = $(\{q\}, \{(q, \{a, b\})\}, P(\{a, b, c\}))$, $\mathcal{F}^{-1}\mathcal{C}$ = $(\{p\}, \{(p, □)\}, P(\{x, y, z\}))$, $\mathcal{F}^{-1}\mathcal{D}$ = $(\{p\}, \{(p, \{x, y, z\})\}, P(\{x, y, z\}))$, $\mathcal{F}^{-1}\mathcal{C} \cap \mathcal{F}^{-1}\mathcal{D}$ = $(\{p\}, \{(p, □)\}, P(\{x, y, z\}))$, $\mathcal{F}^{-1}(\mathcal{C} \cap \mathcal{D})$ = $(\{p\}, \{(p, \{y, z\})\}, P(\{x, y, z\}))$, implying $\mathcal{F}^{-1}(\mathcal{C} \cap \mathcal{D}) \neq \mathcal{F}^{-1}\mathcal{C} \cap \mathcal{F}^{-1}\mathcal{D}$.

Here onwards all our s-maps are the ones defined as above.

(t) Observe that whenever, $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an s-map, F is onto and extended. Consequently, (1) for all $B \in P(U_B)$, $\cup F^{-1}B$ = $(F|U)^{-1}B = F_0^{-1}B$. (2) for all $C \subseteq U_A$, the image of the element $C \in P(U_A)$ under F is the same as the image of the set C under F_0 or $FC = F_0C$ (3) $\mathcal{F}$ is increasing iff $\overline{B}fa \supseteq F\overline{A}a$ for all a in A iff $\overline{B}fa \supseteq F_0\overline{A}a$ for all a in A, $\mathcal{F}$ is decreasing iff $\overline{B}fa \subseteq F\overline{A}a$ for all a in A iff $\overline{B}fa \subseteq F_0\overline{A}a$ for all a in A, $\mathcal{F}$ is preserving iff $\overline{B}fa = F\overline{A}a$ for all a in A iff $\overline{B}fa = F_0\overline{A}a$ for all a in A.

(u) Hence, for any s-map $\mathcal{F}: \mathcal{A} \to \mathcal{B}$,

(i) $\mathcal{F}$ is *increasing*, denoted by $\mathcal{F}_i$ iff $\overline{B}fa \supseteq F_0\overline{A}a$ for all a in A

(ii) $\mathcal{F}$ is *decreasing*, denoted by $\mathcal{F}_d$ iff $\overline{B}fa \subseteq F_0\overline{A}a$ for all a in A

(iii) $\mathcal{F}$ is *preserving*, denoted by $\mathcal{F}_p$ iff $\overline{B}fa = F_0\overline{A}a$ for all a in A.

For any s-map $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, (defined as in (t) above)

(v) for any s-subset $\mathcal{C}$ of $\mathcal{A}$, the *s-image* of $\mathcal{C}$ under $\mathcal{F}$, denoted by $\mathcal{F}\mathcal{C}$, is defined by the s-set $\mathcal{D}$, where (i) $D = fC$ (ii) U_D = F_0U_C or $PU_D = (F_0PU_C)_{P(U_B)} = P(F_0U_C) = (F_0U_C)_{P(U_B)}$ (iii) $\overline{D}: D \to PU_D$ is given by $\overline{D}d = \overline{B}d \cap \cup F\overline{C}(f^{-1}d \cap C)$ for all $d \in D$

(w) for any s-subset $\mathcal{D}$ of $\mathcal{B}$, the *s-inverse image* of $\mathcal{D}$ under $\mathcal{F}$, denoted by $\mathcal{F}^{-1}\mathcal{D}$, is defined by the s-set $\mathcal{C}$, where (i) C = $f^{-1}(D)$ (ii) $U_C = F_0^{-1}(U_D)$ or $P(U_C) = F_0^{-1}(P(U_D)) = P(F_0^{-1}(U_D))$ (iii) $\overline{C}: C \to P(U_C)$ is given by $\overline{C}c = \overline{A}c \cap \cup F^{-1}\overline{D}fc$ for all $c \in C$.

(E) **s-Groups and s-Homomorphisms of s-Groups** In this section we recall the notions of s-group, s-(normal) subgroups, s-product, s-quotient groups etc. from Murthy-Gouthami[12] .

(a) An s-set $\mathcal{G}$ is said to be an *s-(normal) group* iff (i) U_G is a group (ii) $\bar{G}g$ is a (normal) subgroup of U_G for all $g \in G$. An s-group which is also a whole s-set is a *whole s-group*. Clearly, whole s-group and whole s-normal group are the same.

(b) For any s-group $\mathcal{G}$ and for any s-subset $\mathcal{A}$ of $\mathcal{G}$,

(1) $\mathcal{A}$ is an *s-subgroup* of $\mathcal{G}$ iff (i) $A \subseteq G$ (ii) U_A is a subgroup of U_G (iii) $\bar{A}g$ is a subgroup of $\bar{G}g$ for all $g \in A$.
Notice that, a degenerated s-subset $\mathcal{A}$ of an s-group $\mathcal{G}$ is an s-subgroup iff U_A is a subgroup of U_G.
An s-subgroup $\mathcal{A}$ is an *identity s-subgroup* of $\mathcal{G}$ iff $U_A = (e_{U_G})$ and $\bar{A}g = (e_{U_G})$ for all $g \in A$.

(2) $\mathcal{A}$ is an *s-normal subgroup* of $\mathcal{G}$ iff (i) $A \subseteq G$ (ii) U_A is a normal subgroup of U_G (iii) $\bar{A}g$ is a normal subgroup of $\bar{G}g$ for all $g \in G$.
Notice that, a degenerated s-subset $\mathcal{A}$ of $\mathcal{G}$ is an s-normal subgroup iff U_A is a normal subgroup of U_G and a degenerated s-subset which is also an s-(normal) subgroup is called a *degenerated s-(normal) subgroup*.

(c) For any s-group $\mathcal{G}$ and for any s-subsets $\mathcal{A}, \mathcal{B}$ of $\mathcal{G}$, the *s-product* of $\mathcal{A}$ by $\mathcal{B}$, denoted by $\mathcal{A}\mathcal{B}$, is defined by $\mathcal{C}$, where $C = A \cap B$, $U_C = U_AU_B$ and $\bar{C}c = \bar{A}c\bar{B}c$ for all $c \in C$. If C = □ then $\bar{C}$ = □ or $(□, □, P(U_C)) = \mathcal{C}$.

(d) For any s-group $\mathcal{G}$, and for any s-normal subgroup $\mathcal{N}$ of $\mathcal{G}$, the *s-quotient group*, denoted by $\frac{\mathcal{G}}{\mathcal{N}}$, is defined by $\mathcal{C}$, where $C = N \cap G = N$, $U_C = \frac{U_G}{U_N}$ and $\bar{C}c = \frac{\bar{G}cU_N}{U_N}$ for all $c \in C$. Once again if C = □ then $\bar{C}$ = □ or $(□, □, P(U_C)) = \mathcal{C}$.

(e) For any s-set $\mathcal{A}$ and for all $B \subseteq A$, the *restriction* of $\mathcal{A}$ to B, denoted by $\mathcal{A}|B$, is defined by $\mathcal{C}$ iff $C = B$, $U_C = U_A$, $\bar{C}: C \to P(U_C)$ is given by $\bar{C}b = \bar{A}b$ for all $b \in B$.

In what follows we recall some notions like s-homomorphism, s-isomorphism, pure s-isomorphism, s-kernel etc., from Murthy-Gouthami[13] and some results from the above papers which will be used in the main section.

(f) An s-map $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an *s-homomorphism* of s-groups, again denoted by $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, iff (1) both $\mathcal{A}, \mathcal{B}$ are s-groups (2) $F: P(U_A) \to P(U_B)$ is any map such that $F_0 = F|U_A: U_A \to U_B$ is a group homomorphism.

Note: Since any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an s-map, from the definition of s-map $F = P(F_0)$, for any subset C of U_A, $F(C) = F_0(C)$ and so we know F if we know F_0 and vice versa. Consequently, in all our examples we specify *only* $f: A \to B$ and $F_0: U_A \to U_B$ from which follow the s-map $\mathcal{F} = (f, F)$.

(g) An s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an *s-monomorphism* of s-groups, again denoted by $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, iff both f is one-one and F is one-one.

(h) An s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an *s-epimorphism* of s-groups, again denoted by $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, iff both f is onto and F is onto.

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(i) An s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is an *s-isomorphism* of s-groups, again denoted by $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, iff (1) $F_0: U_A \to U_B$ is an isomorphism (2) $F_0|\bar{A}a: \bar{A}a \to \bar{B}fc$ is an isomorphism for all $c \in f^{-1}fa$ and for all $a \in A$ (3) $\mathcal{F}$ is preserving.
(j) An s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is a *pure s-isomorphism* of s-groups, again denoted by $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, iff (1) $f: A \to B$ is a bijection (2) $F_0: U_A \to U_B$ is an isomorphism (3) $F_0|\bar{A}c: \bar{A}c \to \bar{B}fc$ is an isomorphism (4) $\mathcal{F}$ is preserving.
Clearly, (1) any pure s-isomorphism is an s-isomorphism but *not* conversely. (2) For any $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, if $A =$ ⍰ then F_0 is a crisp group isomorphism iff $\mathcal{F}$ is s-isomorphism. However, (3) $\mathcal{F}$ is pure s-isomorphism iff F_0 is a crisp group isomorphism and $B =$ ⍰.
(k) For an s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups, s-*Kernel* of $\mathcal{F}$, denoted by $Ker(\mathcal{F})$, is defined by the whole s-set $Ker(\mathcal{F}) = \mathcal{K}$, where $K = A$, $P(U_K) = P(Ker(F_0))$ or $U_K = Ker(F_0)$ and $\bar{K}: K \to P(U_K)$ is given by $\bar{K}k = Ker(F_0)$ for all $k \in A$.
(l) For any s-map $\mathcal{F}: \mathcal{A} \to \mathcal{B}$, an s-subset $\mathcal{C}$ of $\mathcal{A}$ is an $\mathcal{F}$-*constant* or *constant on each kernel class* iff $\bar{C}a = \bar{C}c$ for all $a \in f^{-1}fc$ for all $c \in C$.
(m) For any s-map $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ and for any s-subset $\mathcal{D}$ of $\mathcal{B}$, $\mathcal{F}^{-1}\mathcal{D}$ is always an $\mathcal{F}$-constant subset of $\mathcal{A}$, whenever $\mathcal{A}$ is $\mathcal{F}$-constant.
(n) For any pair of s-(normal) subgroups $\mathcal{A}, \mathcal{B}$ of $\mathcal{G}$, $\mathcal{A}$ is an s-(normal) subgroup of $\mathcal{B}$ iff $\mathcal{A}$ is an s-subset of $\mathcal{B}$.
(o) For any s-group $\mathcal{G}$, for any s-subgroup $\mathcal{A}$ of $\mathcal{G}$ and for any s-normal subgroup $\mathcal{B}$ of $\mathcal{G}$ we have $\mathcal{A} \cap \mathcal{B}$ is an s-normal subgroup of $\mathcal{A}$.
(p) For any s-group $\mathcal{G}$, for any s-subset $\mathcal{H}$ of $\mathcal{G}$ and for any s-subgroup $\mathcal{K}$ of $\mathcal{G}$ such that $\mathcal{H} \subseteq \mathcal{K}$, we have $\mathcal{H}$ is an s-subgroup of $\mathcal{K}$ iff $\mathcal{H}$ is an s-subgroup of $\mathcal{G}$.
(q) For any s-group $\mathcal{G}$ and for any pair of s-subsets $\mathcal{H}, \mathcal{K}$ of $\mathcal{G}$ such that $\mathcal{H} \subseteq \mathcal{K}$ and $\mathcal{K}$ is an s-subgroup of $\mathcal{G}$, we have $\mathcal{H}$ is an s-normal subgroup of $\mathcal{G}$ implies $\mathcal{H}$ is an s-normal subgroup of $\mathcal{K}$.
(r) For any s-group $\mathcal{G}$ and for any pair of s-(normal) subgroups $\mathcal{H}, \mathcal{K}$ of $\mathcal{G}$, $\mathcal{HK}$ is an s-(normal) subgroup of $\mathcal{G}$ iff $\mathcal{HK} = \mathcal{KH}$.
(s) For any s-group $\mathcal{G}$ and for any pair of s-subgroups $\mathcal{H}, \mathcal{K}$ of $\mathcal{G}$ such that $\mathcal{H}$ or $\mathcal{K}$ is an s-normal subgroup of $\mathcal{G}$ we have $\mathcal{HK}$ is an s-subgroup of $\mathcal{G}$.
(t) For any s-normal subgroup $\mathcal{A}$ of an s-group $\mathcal{G}$ and for any s-(normal) subgroup $\mathcal{B}$ of $\mathcal{G}$ such that $\mathcal{A} \subseteq \mathcal{B}$, $\frac{\mathcal{B}}{\mathcal{A}}$ is an s-(normal) subgroup of $\frac{\mathcal{G}}{\mathcal{A}}$.
(u) For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups and for any s-subgroup $\mathcal{C}$ of $\mathcal{A}$, the s-image $\mathcal{FC}$ of $\mathcal{C}$ under $\mathcal{F}$ is an s-subgroup of $\mathcal{B}$, whenever $\mathcal{C}$ is constant on each kernel class. Further, whenever $\mathcal{A}$ is also constant on each kernel class, $\mathcal{FC}$ is an s-subgroup of $\mathcal{FA}$.
(v) For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups and for any s-normal subgroup $\mathcal{C}$ of $\mathcal{A}$ the s-image $\mathcal{FC}$ of $\mathcal{C}$ under $\mathcal{F}$ is an s-normal subgroup of $\mathcal{FA}$, whenever both $\mathcal{A}$ and $\mathcal{C}$ are constants on each kernel class.
(w) For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups and for any s-subgroup $\mathcal{D}$ of $\mathcal{B}$, the s-inverse image, $\mathcal{F}^{-1}\mathcal{D}$ of $\mathcal{D}$ under $\mathcal{F}$ is an s-subgroup of both $\mathcal{A}$ and $\mathcal{F}^{-1}B$.
(x) For any s-homomorphism $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ of s-groups and for any s-normal subgroup $\mathcal{D}$ of $\mathcal{B}$, the s-inverse image $\mathcal{F}^{-1}\mathcal{D}$ of $\mathcal{D}$ under $\mathcal{F}$ is an s-normal subgroup of $\mathcal{F}^{-1}\mathcal{B}$.
(y) For any s-epimorphism, $\mathcal{F}_d: \mathcal{A} \to \mathcal{B}$ of s-groups, $\mathcal{F}_d\mathcal{A} = \mathcal{B}$.
(z) For any s-homomorphism $\mathcal{F}_i: \mathcal{A} \to \mathcal{B}$ of s-groups, $\mathcal{F}_i^{-1}\mathcal{B} = \mathcal{A}$.

III. MAIN THEOREMS

As mentioned earlier in the Introduction, in Murthy-Gouthami[12] the notion of soft (normal sub, sub) group is generalized to that of an s-(normal sub, sub) group and several of the crisp group theoretic properties of (inverse) images were shown to have extended and in Murthy-Gouthami[13] the notion of homomorphism of groups in the crisp set up is generalized to that of an s-homomorphism of s-groups and several of the crisp group homomorphic properties were shown to have extended. Now in this section we generalize and extend the correspondence and the three Isomorphism theorems of groups in the crisp set up to those of s-groups.

Correspondence Theorem for s-(normal) subgroups
Theorem 3.1 *For any s-epimorphism $\mathcal{F}_p: \mathcal{A} \to \mathcal{B}$ of s-groups there is a one-to-one correspondence between the set of all s-(normal) subgroups $\mathcal{C}$ of $\mathcal{A}$ which are constant on each kernel class and which contain $Ker(\mathcal{F})$ and the set of all s-(normal) subgroups of $\mathcal{FA} = \mathcal{B}$.*
Proof: Let $\mathbb{I}$ be the set of all s-(normal) subgroups $\mathcal{C}$ of $\mathcal{A}$ which are constant on each kernel class and which contain $Ker(\mathcal{F})$ and let $\mathbb{J}$ be the set of all s-(normal) subgroups of $\mathcal{B}$.
Define $\phi: \mathbb{I} \to \mathbb{J}$ by $\phi(\mathcal{C}) = \mathcal{F}_p(\mathcal{C})$. Then since every $\mathcal{C}$ in $\mathbb{I}$ is constant on each kernel class, by 2(E)(u),(v) $\phi(\mathcal{C}) = \mathcal{F}_p(\mathcal{C})$ is an s-(normal) subgroup of $\mathcal{B}$ and so ϕ is well-defined.

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Define $\psi: \mathbb{J} \to \mathbb{I}$ by $\psi(\mathcal{D}) = \mathcal{F}_p^{-1}(\mathcal{D})$. Then by 2(E)(w),(x) and (z) $\psi(\mathcal{D}) = \mathcal{F}_p^{-1}(\mathcal{D})$ is an s-(normal) subgroup of $\mathcal{A}$ and so ψ is well-defined.
To show ϕ defines a one-one correspondence between $\mathbb{I}$ and $\mathbb{J}$, it is enough to show $\psi \circ \phi = 1_{\mathbb{I}}$ and $\phi \circ \psi = 1_{\mathbb{J}}$ or equivalently $\mathcal{F}_p^{-1}(\mathcal{F}_p(\mathcal{C})) = \mathcal{C}$ and $\mathcal{F}_p(\mathcal{F}_p^{-1}\mathcal{D}) = \mathcal{D}$.
In what follows, we show that for any $\mathcal{C} \in \mathbb{I}$, $\mathcal{F}_p^{-1}(\mathcal{F}_p(\mathcal{C})) = \mathcal{C}$.
Let $\mathcal{F}\mathcal{C}$=$\mathcal{P}$. Then P=fC, U_P=F_0U_C and $\overline{P}p = \overline{B}p \cap (\cup_{c\in f^{-1}p\cap C} F_0\overline{C}c)$ for all $p \in P$.
Let $\mathcal{F}^{-1}\mathcal{P} = \mathcal{Q}$. Then $Q = f^{-1}P$, $U_Q = F_0^{-1}U_P$ and $\overline{Q}q = \overline{A}q \cap F_0^{-1}\overline{P}fc$ for all $q \in Q$.
$Ker(\mathcal{F}) \subseteq \mathcal{C}$ implies $K = C = A$, $U_K = Ker(F_0) \subseteq U_C$ and $\overline{K}k = Ker(F_0) \subseteq \overline{C}k$ for all $k \in A$.
We claim that $\mathcal{Q} = \mathcal{C}$ or (i) $Q = C$ (ii) $U_Q = U_C$ and (iii) $\overline{Q}q = \overline{C}q$ for all $q \in Q$
(i): $Q = f^{-1}P = f^{-1}fC = f^{-1}(fA) = A = C$.
(ii): $U_Q = F_0^{-1}U_P = F_0^{-1}(F_0U_C) = U_C$, where the last equality is due to the fact that $Ker(F_0) \subseteq U_C$ and by 2(A)(ii).
(iii): Let $q \in Q = C$ be fixed. Then $\overline{Q}q = \overline{A}q \cap F_0^{-1}\overline{P}fq = \overline{A}q \cap F_0^{-1}(\overline{B}fq \cap (\cup_{c\in f^{-1}fq\cap C} F_0\overline{C}c)) = \overline{A}q \cap F_0^{-1}\overline{B}fq \cap F_0^{-1}(\cup_{c\in f^{-1}fq\cap C} F_0\overline{C}c)$.
Since $\mathcal{C}$ is constant on each kernel class, $\overline{C}c = \overline{C}q$ for all $c \in f^{-1}fq \cap C$ implies $\cup_{c\in f^{-1}fq\cap C} F_0\overline{C}c = F_0\overline{C}q$ which implies $\overline{Q}q = \overline{A}q \cap F_0^{-1}\overline{B}fq \cap F_0^{-1}(F_0\overline{C}q) \overset{(2)}{=} \overline{A}q \cap F_0^{-1}\overline{B}fq \cap \overline{C}q \overset{(3)}{=} F_0^{-1}\overline{B}fq \cap \overline{C}q$, where the second equality is due to the fact that $Ker(F_0) \subseteq \overline{C}q$ and the third equality is due to $\mathcal{C} \subseteq \mathcal{A}$.
Now since $\mathcal{F}$ is preserving and $\mathcal{C}$ is an s-subgroup of $\mathcal{A}$, $F_0\overline{C}q \subseteq F_0\overline{A}q = \overline{B}fq$ or $F_0\overline{C}q \subseteq \overline{B}fq$ which implies $\overline{C}q \subseteq F_0^{-1}\overline{B}fq$ or $\overline{C}q \cap F_0^{-1}\overline{B}fq = \overline{C}q$. Therefore $\overline{Q}q = \overline{C}q \cap F_0^{-1}\overline{B}fq = \overline{C}q$ or $\mathcal{C} = \mathcal{Q}$.

In what follows, we show that for any $\mathcal{D} \in \mathbb{J}$, $\mathcal{F}_p(\mathcal{F}_p^{-1}(\mathcal{D})) = \mathcal{D}$.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{R}$. Then $R = f^{-1}D$, $U_R = F_0^{-1}U_D$ and $\overline{R}r = \overline{A}r \cap F_0^{-1}\overline{D}fr$ for all $r \in R$.
Let $\mathcal{F}\mathcal{R} = \mathcal{S}$. Then $S = fR$, $U_S = F_0U_R$ and $\overline{S}s = \overline{B}s \cap (\cup_{c\in f^{-1}s\cap R} F_0\overline{R}r)$ for all $s \in S$.
We show that $\mathcal{S} = \mathcal{D}$ or (i) $S = D$ (ii) $U_S = U_D$ and (iii) $\overline{S}s = \overline{D}s$ for all $s \in S$.
(i): $S = fR = ff^{-1}D = D$, where the last equality is due to f is onto.
(ii): $U_S = F_0U_R = F_0(F_0^{-1}U_D) = U_D$, where the last equality is due to F_0 is onto.
(iii): Let $s \in S = D$ be fixed and $r \in f^{-1}s \cap R$. Then $s = fr$, $\overline{S}s = \overline{B}s \cap (\cup_{r\in f^{-1}s\cap R} F_0\overline{R}r)$. Now we show that $F_0\overline{R}r = \overline{D}s$ for all $r \in f^{-1}s \cap R$.
(a) Since $r \in f^{-1}s$, $fr = s$. Since $\mathcal{D} \subseteq \mathcal{B}$, $\mathcal{F}$ is preserving we have $\overline{D}s = \overline{D}fr \subseteq \overline{B}fr = F_0\overline{A}r$ which implies $\overline{D}fr \cap F_0\overline{A}r = \overline{D}fr$.
Now $F_0\overline{R}r = F_0(\overline{A}r \cap F_0^{-1}\overline{D}fr) \subseteq F_0\overline{A}r \cap F_0F_0^{-1}\overline{D}fr = F_0\overline{A}r \cap \overline{D}fr = \overline{D}fr = \overline{D}s$ which implies $F_0\overline{R}r \subseteq \overline{D}s$.
(b) Since $\mathcal{D} \subseteq \mathcal{B}$, $\overline{D}s = \overline{D}fr \subseteq \overline{B}fr = F_0\overline{A}r$ or $\overline{D}s = \overline{D}s \cap F_0\overline{A}r$ and since F_0 is onto, we have $F_0F_0^{-1}\overline{D}s = \overline{D}s = \overline{D}s \cap F_0\overline{A}r$ which implies $\overline{D}s = F_0F_0^{-1}\overline{D}s \cap F_0\overline{A}r$. Let $\beta \in \overline{D}s = F_0F_0^{-1}\overline{D}s \cap F_0\overline{A}r$ which implies $\beta = F_0\alpha$, $\alpha \in F_0^{-1}\overline{D}s$, $\beta = F_0\gamma$, $\gamma \in \overline{A}r$ which implies $F_0\alpha = F_0\gamma$ which implies $\alpha - \gamma \in ker(F_0) = F_0^{-1}(0) \subseteq F_0^{-1}\overline{D}s$ with $\alpha \in F_0^{-1}\overline{D}s$ which implies $\gamma = \gamma - \alpha + \alpha \in F_0^{-1}\overline{D}s$ which implies $\gamma \in F_0^{-1}\overline{D}s \cap \overline{A}r = \overline{R}r$ which implies $\beta = F_0\gamma \in F_0\overline{R}r$ which in turn implies $\overline{D}s \subseteq F_0\overline{R}r$.
From (a) and (b) we get $F_0\overline{R}r = \overline{D}s$ for all $r \in f^{-1}s \cap R$.
Therefore, $\cup_{r\in f^{-1}s\cap R} F_0\overline{R}r = \cup_{r\in f^{-1}s\cap R} \overline{D}s = \overline{D}s$, implying $\overline{S}s = \overline{B}s \cap \overline{D}s = \overline{D}s$ or $\mathcal{S} = \mathcal{D}$.

The following Example shows that the above Theorem is *not* true if $\mathcal{F}$ is decreasing, $\mathcal{F}$ is onto, $Ker(\mathcal{F}) \subseteq \mathcal{C}$ but $\mathcal{C}$ is constant on each kernel class.
Example 2: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A}$=$(\{a\}, \{(a, \mathbb{Z}_4)\}, P(\mathbb{Z}_4))$=$\mathcal{C}$, $\mathcal{B}$=$(\{b\}, \{(b, (0))\}, P(\mathbb{Z}_4))$, $f: A \to B$ given by $f = \{(a, b)\}$ and F_0 be the identity map.
Then $\overline{B}fa = (0) \subseteq \mathbb{Z}_4 = F_0\overline{A}a$, implying $\mathcal{F}$ is decreasing, $Ker(\mathcal{F}) = \mathcal{K}$=$(\{a\}, \{(a, (0))\}, P(0)) \subseteq \mathcal{C}$, $\mathcal{F}$ is onto and $\mathcal{C}$ is constant on each kernel class because $\overline{C}a = \mathbb{Z}_4$.
Let $\mathcal{F}\mathcal{C} = \mathcal{D}$. Then $D = fC = \{b\}$, $U_D = F_0U_C = \mathbb{Z}_4$ and $\overline{D}b = \overline{B}b \cap (\cup_{c\in f^{-1}b\cap C} F_0\overline{C}c) = (0) \cap \mathbb{Z}_4 = (0)$.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{E}$. Then $E = f^{-1}D = \{a\}$, $U_E = F_0^{-1}U_D = \mathbb{Z}_4$ and $\overline{E}a = \overline{A}a \cap F_0^{-1}\overline{D}fa = \mathbb{Z}_4 \cap (0) = (0) \neq \mathbb{Z}_4 = \overline{C}a$ or $\mathcal{F}_d^{-1}\mathcal{F}_d\mathcal{C} \neq \mathcal{C}$.

The following Example shows that the above Theorem is *not* true if $\mathcal{F}$ is increasing, $\mathcal{F}$ is onto.
Example 3: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a\}, \{(a, Z_2)\}, P(\mathbb{Z}_4))$, $\mathcal{B} = (\{b\}, \{(b, (\mathbb{Z}_4))\}, P(\mathbb{Z}_4)) = \mathcal{D}$, $f: A \to B$ given by $f = \{(a, b)\}$ and F_0 be the identity map.
Then $\overline{B}fa = \mathbb{Z}_4 \supseteq Z_2 = F_0\overline{A}a$, implying $\mathcal{F}$ is increasing, $Ker(\mathcal{F}) = \mathcal{K} = (\{a\}, \{(a, (\bar{0}))\}, P(\bar{0})) \subseteq \mathcal{D}$, $\mathcal{F}$ is onto.

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Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{C}$. Then $C = f^{-1}D = \{a\}$, $U_C = F_0^{-1}U_D = \mathbb{Z}_4$ and $\overline{C}a = \overline{A}a \cap F_0^{-1}\overline{D}fa = Z_2 \cap \mathbb{Z}_4 = Z_2$. $Ker(\mathcal{F}) \subseteq \mathcal{C}$.
Let $\mathcal{F}\mathcal{C} = \mathcal{E}$. Then $E = fC = \{b\}$, $U_E = F_0U_C = \mathbb{Z}_4$ and $\overline{E}b = \overline{B}b \cap (\cup_{c\in f^{-1}b\cap C} F_0\overline{C}c) = \mathbb{Z}_4 \cap Z_2 = Z_2 \neq \mathbb{Z}_4 = \overline{D}b$ or $\mathcal{F}_i\mathcal{F}_i^{-1}\mathcal{D} \neq \mathcal{D}$.

The following Example shows that the above Theorem is *not* true if $\mathcal{F}$ is preserving, f is *not* onto, F_0 is onto.
Example 4: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a_1, a_2\}, \{(a_1, Z_2), (a_2, Z_2)\}, P(\mathbb{Z}_4))$, $\mathcal{B} = (\{b_1, b_2\}, \{(b_1, Z_2), (b_2, Z_2)\}, P(\mathbb{Z}_4)) = \mathcal{D}$, $f: A \to B$ given by $f = \{(a_1, b_1), (a_2, b_1)\}$ and F_0 be the identity map.
Then $\overline{B}fa_1 = Z_2 = F_0\overline{A}a_1$ and $\overline{B}fa_2 = Z_2 = F_0\overline{A}a_2$, implying $\mathcal{F}$ is preserving, $Ker(\mathcal{F}) = \mathcal{K} = (\{a_1, a_2\}, \{(a_1, (0)), (a_2, (0))\}, P(0))$, f is *not* onto, F_0 is onto.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{C}$. Then $C = f^{-1}D = \{a_1, a_2\}$, $U_C = F_0^{-1}U_D = \mathbb{Z}_4$ and $\overline{C}a_1 = \overline{A}a_1 \cap F_0^{-1}\overline{D}fa_1 = Z_2 \cap Z_2 = Z_2 = \overline{C}a_2$. $Ker(\mathcal{F}) \subseteq \mathcal{C}$.
Let $\mathcal{F}\mathcal{C} = \mathcal{E}$. Then $E = fC = \{b_1\} \neq \{b_1, b_2\} = D$, implying $\mathcal{F}_p\mathcal{F}_p^{-1}\mathcal{D} \neq \mathcal{D}$.

The following Example shows that the above Theorem is *not* true if $\mathcal{F}$ is preserving, f is onto, F_0 is *not* onto.
Example 5: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a\}, \{(a, \mathbb{Z}_2)\}, P(\mathbb{Z}_2))$, $\mathcal{B} = (\{b\}, \{(b, Z_2)\}, P(\mathbb{Z}_4)) = \mathcal{D}$, $f: A \to B$ given by $f = \{(a, b)\}$ and F_0 be the inclusion map $F_0 = \{(\overline{0}, \overline{0}), (\overline{1}, \overline{2})\}$.
Then $\overline{B}fa = Z_2 = F_0\overline{A}a$, implying $\mathcal{F}$ is preserving, $Ker(\mathcal{F}) = \mathcal{K} = (\{a\}, \{(a, (0))\}, P(0))$, F_0 is *not* onto.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{C}$. Then $C = f^{-1}D = \{a\}$, $U_C = F_0^{-1}U_D = \mathbb{Z}_2$ and $\overline{C}a = \overline{A}a \cap F_0^{-1}\overline{D}fa = \mathbb{Z}_2 \cap \mathbb{Z}_2 = \mathbb{Z}_2$. $Ker(\mathcal{F}) \subseteq \mathcal{C}$.
Let $\mathcal{F}\mathcal{C} = \mathcal{E}$. Then $E = fC = \{b\}$, $U_E = F_0U_C = Z_2 \neq \mathbb{Z}_4 = U_D$, implying $\mathcal{F}_p\mathcal{F}_p^{-1}\mathcal{D} \neq \mathcal{D}$.

The following Example shows that the above Theorem is *not* true if $\mathcal{F}$ is preserving, $\mathcal{F}$ is onto, $Ker(\mathcal{F}) \nsubseteq \mathcal{C}$ but $\mathcal{C}$ is constant on each kernel class.
Example 6: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a\}, \{(a, \mathbb{Z}_4)\}, P(\mathbb{Z}_4))$, $\mathcal{B} = (\{b\}, \{(b, Z_2)\}, P(\mathbb{Z}_4))$, $\mathcal{C} = (\{a\}, \{(a, (0))\}, P(\mathbb{Z}_4))$, $f: A \to B$ given by $f = \{(a, b)\}$ and $F_0 = \{(0,0), (2,0), (1,1), (3,1)\}$.
Then $\overline{B}fa = Z_2 = F_0\overline{A}a$, implying $\mathcal{F}$ is preserving, $Ker(\mathcal{F}) = \mathcal{K} = (\{a\}, \{(a, Z_2)\}, P(Z_2)) \nsubseteq \mathcal{C}$, $\mathcal{F}$ is onto and $\mathcal{C}$ is constant on each kernel class because $\overline{C}a = (0)$.
Let $\mathcal{F}\mathcal{C} = \mathcal{D}$. Then $D = fC = \{b\}$, $U_D = F_0U_C = Z_2$ and $\overline{D}b = \overline{B}b \cap (\cup_{c\in f^{-1}b\cap C} F_0\overline{C}c) = Z_2 \cap (0) = (0)$.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{E}$. Then $E = f^{-1}D = \{a\}$, $U_E = F_0^{-1}U_D = \mathbb{Z}_4$ and $\overline{E}a = \overline{A}a \cap F_0^{-1}\overline{D}fa = \mathbb{Z}_4 \cap Z_2 = Z_2 \neq (0) = \overline{C}a$ or $\mathcal{F}_p^{-1}\mathcal{F}_p\mathcal{C} \neq \mathcal{C}$.

The following Example shows that the above Theorem is *not* true if $\mathcal{F}$ is preserving, $\mathcal{F}$ is onto, $Ker(\mathcal{F}) \subseteq \mathcal{C}$ but $\mathcal{C}$ is *not* constant on each kernel class.
Example 7: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by: $\mathcal{A} = (\{a_1, a_2\}, \{(a_1, \mathbb{Z}), (a_2, \mathbb{Z})\}, P(\mathbb{Z}))$, $\mathcal{B} = (\{b\}, \{(b, \mathbb{Z})\}, P(\mathbb{Z}))$, $\mathcal{C} = (\{a_1, a_2\}, \{(a_1, 2\mathbb{Z}), (a_2, 3\mathbb{Z})\}, P(\mathbb{Z}))$, $f: A \to B$ given by $f = \{(a_1, b), (a_2, b)\}$ and F_0 be the identity map.
Then $\overline{B}fa_1 = \mathbb{Z} = F_0\overline{A}a_1$ and $\overline{B}fa_2 = \mathbb{Z} = F_0\overline{A}a_2$, implying $\mathcal{F}$ is preserving, $Ker(\mathcal{F}) = \mathcal{K} = (\{a_1, a_2\}, \{(a_1, (0)), (a_2, (0))\}, P(0)) \subseteq \mathcal{C}$, $\mathcal{F}$ is onto and $\mathcal{C}$ is *not* constant on each kernel class because $\overline{C}a_1 = 2\mathbb{Z}$, $\overline{C}a_2 = 3\mathbb{Z}$.
Let $\mathcal{F}\mathcal{C} = \mathcal{D}$. Then $D = fC = \{b\}$, $U_D = F_0U_C = \mathbb{Z}$ and $\overline{D}b = \overline{B}b \cap (\cup_{c\in f^{-1}b\cap C} F_0\overline{C}c) = \mathbb{Z} \cap (2\mathbb{Z} \cup 3\mathbb{Z}) = 2\mathbb{Z} \cup 3\mathbb{Z}$.
Let $\mathcal{F}^{-1}\mathcal{D} = \mathcal{E}$. Then $E = f^{-1}D = \{a_1, a_2\}$, $U_E = F_0^{-1}U_D = \mathbb{Z}$ and $\overline{E}a_1 = \overline{A}a_1 \cap F_0^{-1}\overline{D}fa_1 = \mathbb{Z} \cap (2\mathbb{Z} \cup 3\mathbb{Z}) = 2\mathbb{Z} \cup 3\mathbb{Z} = \overline{E}a_2$.
Clearly, $\mathcal{F}\mathcal{C} = \mathcal{D}$ and $\mathcal{F}^{-1}\mathcal{D} = \mathcal{E}$ are *not* even s-subgroups.

First Isomorphism Theorem for *s*-groups
Theorem 3.2 *For any s-homomorphism $\mathcal{F}_i: \mathcal{A} \to \mathcal{B}$ of s-groups such that $Ker(\mathcal{F}) \subseteq \mathcal{A}$, $\frac{\mathcal{A}}{Ker(\mathcal{F})}$ is isomorphic to $\mathcal{F}\mathcal{A}$, whenever $\mathcal{A}$ is constant an each kernel class.*
Proof: Let $\mathcal{K} = Ker(\mathcal{F}) \subseteq \mathcal{A}$. Then $K = A$, $U_K = Ker(F_0) \subseteq U_A$ and $\overline{K}k = Ker(F_0) \subseteq \overline{A}k$ for all $k \in K$.
Let $\frac{\mathcal{A}}{\mathcal{K}} = \mathcal{C}$. Then $C = A$, $U_C = \frac{U_A}{U_K} = \frac{U_A}{Ker(F_0)}$ and $\bar{C}c = \frac{\bar{A}c\ Ker(F_0)}{Ker(F_0)} = \frac{\bar{A}c}{U_K}$.
Since U_C is a group and $\bar{C}c = \frac{\bar{A}c}{U_K}$ is a subgroup of $\frac{U_A}{U_K} = U_C$ as $\bar{A}c$ is a subgroup of U_A, by the definition of an s-group, $\mathcal{C}$ is an s-group.
Let $\mathcal{F}\mathcal{A} = \mathcal{D}$. Then $D = fA$, $U_D = F_0U_A$ and $\bar{D}d = \bar{B}d \cap (\cup_{c\in f^{-1}d\cap A} F_0\bar{A}c)$ for all $d \in D$.
Since image of a subgroup is a subgroup and intersection of subgroups is a subgroup, it follows that U_D is a group and $\bar{D}d$ is a subgroup of U_D. Consequently, $\mathcal{D}$ is an s-group as $\mathcal{A}$ is constant on each kernel class.
We show that $\mathcal{G} = (g, G)$, where $g: C \to D$, $G: P(U_C) \to P(U_D)$ is an s-map such that $G_0: U_C \to U_D$ is the usual isomorphism between $\frac{U_K}{Ker(F_0)}$ and $F_0(U_A)$.

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(i) $g: C \to D$ be the same as f but restricted to the range fA, given by $g: A = C \to D = fA$ defined by $g(a) = f(a)$ for all $a \in C$.
(ii) Let $G_0: U_C = \frac{U_A}{U_K} \to U_D = F_0U_A$ defined by $G_0(U_Kx) = F_0x$ for all $x \in U_A$ be the isomorphism in crisp group theory. Then by 2(D)(t), $F: P(U_C) \to P(U_D)$ is defined by $F(A) = F_0(A)$ for all $A \in P(U_C)$
(iii) Now we show that $\mathcal{G}$ is an s-isomorphism.
Clearly, $g: C \to D$ and $G: P(U_C) \to P(U_D)$ are maps such that $G_0: U_C \to U_D$ is an isomorphism.
It is enough to show that $\bar{D}gc = G_0\bar{C}c$ and $G_0|\bar{C}c: \bar{C}c \to \bar{D}gc$ is an isomorphism.
Let $c \in C$, $\bar{D}gc = \bar{B}gc \cap (\cup_{a\in g^{-1}gc\cap A} F_0\bar{A}a) = \bar{B}fc \cap (\cup_{a\in f^{-1}fc\cap A} F_0\bar{A}a)$. Since $\mathcal{A}$ is constant on each kernel class, $\bar{A}a = \bar{A}c$ for all $a \in f^{-1}fc$ implies $\cup_{a\in f^{-1}fc} F_0\bar{A}a = F_0\bar{A}c$ then $\bar{D}gc = \overline{B}fc \cap F_0\bar{A}c = F_0\overline{A}c$ as $\mathcal{F}$ is increasing.
Now we claim that $F_0\bar{A}c = G_0\bar{C}c$. Since $G_0(U_Kx) = F_0x$ for all $x \in U_A$ and $U_K \subseteq \bar{A}c$, we get $G_0(\frac{\bar{A}c}{U_K}) = F_0(\bar{A}c)$ (or) $G_0\bar{C}c = F_0\bar{A}c$, as $\overline{C}c = \frac{\bar{A}c}{U_K}$. Therefore $\bar{D}gc = G_0\bar{C}c$.
By 2(A)b, observe that $\phi: G \to G'$ is isomorphism of groups and H is a subgroup of G implies $\phi|H: H \to \phi H$ is a group isomorphism.
Hence, since $G_0: \frac{U_A}{U_K} \to F_0(U_A)$ is group isomorphism, $G_0|\bar{C}c = G_0|\frac{\bar{A}c}{U_K}: \frac{\bar{A}c}{U_K} \to G_0(\frac{\bar{A}c}{U_K})$ is a group isomorphism. But $\frac{\bar{A}c}{U_K} = \bar{C}c$ and $G_0(\frac{\bar{A}c}{U_K}) = G_0(\bar{C}c) = \bar{D}gc$. Consequently, $G_0|\bar{C}c: \bar{C}c \to \bar{D}gc$ is an isomorphism.

Corollary 3.3 *For any s-epimorphism $\mathcal{F}_p: \mathcal{A} \to \mathcal{B}$ of s-groups such that $Ker(\mathcal{F}) \subseteq \mathcal{A}$, $\frac{\mathcal{A}}{Ker(\mathcal{F})}$ is isomorphic to $\mathcal{B}$.*
Proof: It follows from 2(E)(y) and 3.2 above.

The following example shows that the above Theorem is *not* true if $\mathcal{F}$ is *not* increasing but $Ker(\mathcal{F}) \subseteq \mathcal{A}$ and $\mathcal{A}$ is constant on each kernel class.
Example 8: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by $\mathcal{A} = (\{a\}, \{(a, \mathbb{Z}_4)\}, P(\mathbb{Z}_4))$, $\mathcal{B} = (\{b\}, \{(b, (0))\}, P(\mathbb{Z}_4))$, $f: A \to B$ be given by $f = \{(a, b)\}$ and F_0 be the identity map.
$\bar{B}fa = \bar{B}b = (0) \subseteq \mathbb{Z}_4 = F_0\bar{A}a$, implying $\mathcal{F}$ is *not* increasing, $Ker(\mathcal{F}) = \mathcal{K} = (\{a\}, \{(a, (0))\}, (\bar{0})\} \subseteq \mathcal{A}$, $\mathcal{A}$ is constant on the kernel class because $f^{-1}fa = f^{-1}\{b\} = \{a\}$, $\bar{A}a = \mathbb{Z}_4$.
Let $\frac{\mathcal{A}}{Ker(\mathcal{F})} = \mathcal{C}$. Then $C = A = \{a\}$, $U_C = \frac{U_A}{Ker(F_0)} = \frac{\mathbb{Z}_4}{(0)} = \mathbb{Z}_4$ and $\bar{C}a = \frac{\bar{A}c}{Ker(F_0)} = \frac{\mathbb{Z}_4}{(0)} = \mathbb{Z}_4$.
Let $\mathcal{F}\mathcal{A} = \mathcal{D}$.Then $D = f\{a\} = \{b\}$, $U_D = F_0U_A = \mathbb{Z}_4$ and $\bar{D}b = \overline{B}d \cap \cup_{c\in f^{-1}d\cap A} F_0\overline{C}c = (0) \cap \mathbb{Z}_4 = (0)$.
Therefore $\bar{D}fa$ is *not* isomorphic to $\bar{C}a$ because $\bar{D}fa = (0)$ is *not* isomorphic to $\mathbb{Z}_4 = \bar{C}a$ or $\mathcal{C}$ is *not* isomorphic to $\mathcal{D}$.

The following example shows that the above Theorem is *not* true if $\mathcal{F}$ is increasing, $Ker(\mathcal{F}) \subseteq \mathcal{A}$ but $\mathcal{A}$ is *not* constant on each kernel class.
Example 9: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be an s-homomorphism given by $\mathcal{A} = (\{a_1, a_2\}, \{(a_1, Z_2), (a_2, \mathbb{Z}_4)\}, P(\mathbb{Z}_4))$, $\mathcal{B} = (\{b\}, \{(b, \mathbb{Z}_4)\}, P(\mathbb{Z}_4))$, $f: A \to B$ be given by $f = \{(a_1, b), (a_2, b)\}$ and F_0 be the identity map.
Then $Ker(\mathcal{F}) = \mathcal{K} = (\{a_1, a_2\}, \{(a_1, (0)), (a_2, (0)), P(0)\}) \subseteq \mathcal{A}$, $\bar{B}fa_1 = \bar{B}b = \mathbb{Z}_4 \supseteq Z_2 = F_0\bar{A}a_1$, $\bar{B}fa_2 = \bar{B}b = \mathbb{Z}_4 = F_0\bar{A}a_2$, implying $\mathcal{F}$ is increasing and $\mathcal{A}$ is *not* constant on each kernel class because $f^{-1}fa_1 = f^{-1}b = \{a_1, a_2\}$, $\bar{A}a_1 = Z_2$, $\bar{A}a_2 = \mathbb{Z}_4$.
Let $\frac{\mathcal{A}}{Ker(F_0)} = \mathcal{C}$. Then $C = A = \{a_1, a_2\}$, $U_C = \frac{U_A}{Ker(F_0)} = \frac{\mathbb{Z}_4}{(0)} = \mathbb{Z}_4$ and $\bar{C}a_1 = \frac{\bar{A}a_1}{Ker(F_0)} = \frac{Z_2}{(0)} = Z_2$, $\bar{C}a_2 = \frac{\bar{A}a_2}{Ker(F_0)} = \frac{\mathbb{Z}_4}{(0)} = \mathbb{Z}_4$.
Let $\mathcal{F}\mathcal{A} = \mathcal{D}$. Then $D = fA = f\{a_1, a_2\} = \{b\}$, $U_D = F_0U_A = F_0\mathbb{Z}_4 = \mathbb{Z}_4$ and $\bar{D}b = \bar{B}b \cap (\cup_{c\in f^{-1}b\cap A} F_0\bar{A}c) = \mathbb{Z}_4 (F_0\bar{A}a_1 \cup F_0\bar{A}a_2) = \mathbb{Z}_4 \cap (Z_2 \cup \mathbb{Z}_4) = \mathbb{Z}_4$.
Clearly, $a_1, a_2 \in f^{-1}fa_2$ but $\bar{C}a_1 = Z_2$ is *not* isomorphic to $\mathbb{Z}_4 = \bar{D}fa_2$ or $\mathcal{C}$ is *not* isomorphic to $\mathcal{D}$.

The following example shows that the above Theorem is *not* true if $\mathcal{F}$ is increasing, $\mathcal{A}$ is constant on each kernel class but $Ker(\mathcal{F}) \nsubseteq \mathcal{A}$
Example 10: Let $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ be given by $\mathcal{A} = (\{a\}, \{(a, Z_2)\}, P(\mathbb{Z}_4))$, $\mathcal{B} = (\{b\}, \{(b, \mathbb{Z}_4)\}, P(\mathbb{Z}_4))$, $f: A \to B$ be given by $f = \{(a, b)\}$ and $F_0: U_A \to U_B$ be given by $F_0 = \{(\mathbb{Z}_4, 0)\}$.
Then $Ker(\mathcal{F}) = \mathcal{K} = (\{a\}, \{(a, \mathbb{Z}_4)\}, P(\mathbb{Z}_4)) \nsubseteq \mathcal{A}$ because $\bar{K}a = \mathbb{Z}_4 \nsubseteq Z_2 = \bar{A}a$, $\bar{B}fa = \bar{B}b = \mathbb{Z}_4 \supseteq (0) = F_0\bar{A}a$, implying $\mathcal{F}$ is increasing, $\mathcal{A}$ is constant on the kernel class, $f^{-1}fa = \{b\}$, $\bar{A}a = Z_2$.
Let $\frac{\mathcal{A}}{Ker(\mathcal{F})} = \mathcal{C}$. Then$C = A = \{a\}$,$U_C$= $\frac{U_A}{Ker(F_0)}$= (0)and $\bar{C}a = \frac{\bar{A}a}{Ker(F_0)} = \frac{Z_2}{\mathbb{Z}_4} = Z_2$, so $\frac{\mathcal{A}}{Ker(\mathcal{F})}$ does *not* exists.
Let $\mathcal{F}\mathcal{A} = \mathcal{D}$.Then$D$=$fA = \{b\}$, $U_D = F_0U_A$ =$F_0\mathbb{Z}_4 = (0)$ and $\bar{D}b = \bar{B}b \cap (\cup_{c\in f^{-1}b\cap A} F_0\bar{A}c) = \mathbb{Z}_4 \cap (0)$=(0).
Therefore $\bar{C}a$ is *not* isomorphic to $\bar{D}fa$ or $\mathcal{C}$ is *not* isomorphic to $\mathcal{D}$.

Second Isomorphism Theorem for s-Groups

Theorem 3.4 *For any s-group $\mathcal{G}$ and for any s-subgroup $\mathcal{A}$ of $\mathcal{G}$ and for any s-normal subgroup $\mathcal{B}$ of $\mathcal{G}$, we have $\frac{\mathcal{A}}{\mathcal{A}\cap\mathcal{B}}$ is purely s-isomorphic to $\frac{\mathcal{A}\mathcal{B}}{\mathcal{B}}$.*

Proof: Let $\mathcal{A}\cap\mathcal{B}=\mathcal{C}$. Then $C=A\cap B$, $U_C=U_A\cap U_B$ and $\bar{C}c=\bar{A}c\cap\bar{B}c$ for all $c\in C$

By 2(E)(o) we have $\mathcal{C}=\mathcal{A}\cap\mathcal{B}$ is a normal subgroup of $\mathcal{A}$, $C\subseteq A$, U_C is a normal subgroup of U_A and $\bar{C}c$ is a normal subgroup of $\bar{A}c$ for all $c\in C$.

Let $\mathcal{A}\mathcal{B}=\mathcal{D}$. Then $D=A\cap B$, $U_D=U_AU_B$ and $\bar{D}d=\bar{A}d\bar{B}d$ for all $d\in D$.

Let $\frac{\mathcal{A}}{\mathcal{C}}=\mathcal{E}$. Then $E=A\cap C=C=A\cap B$, $U_E=\frac{U_A}{U_C}=\frac{U_A}{U_A\cap U_B}$ and $\bar{E}e=\frac{\bar{A}eU_C}{U_C}=\frac{U_C\bar{A}e}{U_C}$ as U_C is a normal subgroup of U_A and $\bar{A}c$ is a subgroup of U_A.

Let $\frac{\mathcal{D}}{\mathcal{B}}=\mathcal{F}$. Then $F=D\cap B=A\cap B\cap B=A\cap B$, $U_F=\frac{U_D}{U_B}=\frac{U_AU_B}{U_B}$ and $\bar{F}f=\frac{\bar{D}fU_B}{U_B}=\frac{U_B\bar{D}f}{U_B}$ as U_B is a normal subgroup of U_G and $\bar{D}fa$ is a subgroup of U_G.

Now we show that there is a pure s-isomorphism $\mathcal{G}=(g,G):\mathcal{F}\to\mathcal{E}$, where $g:F\to E$ and $G:P(U_F)\to P(U_E)$.

First observe that,

$C=$ □ implies$D=$ □,$\overline{C}=$ □ $=\overline{D}$ implies$E=$ □ $=F$,$\overline{E}=$ □, $\overline{F}=$ □ and $U_E=\frac{U_A}{U_A\cap U_B}$ which by second isomorphism theorem of crisp group theory is isomorphic to $\frac{U_AU_B}{U_B}=U_F$ and so by 2(E)(j)(3) $\mathcal{G}$ is a pure s-isomorphism of $\frac{\mathcal{A}}{\mathcal{A}\cap\mathcal{B}}$ onto $\frac{\mathcal{A}\mathcal{B}}{\mathcal{B}}$. Therefore, let $C\neq$ □ $\neq D$.

i) $E=A\cap B=A\cap B\cap B=D\cap B=F$ and g be the identity map of F onto E. Then g is a bijection.

ii) Since $U_F=\frac{U_D}{U_B}=\frac{U_AU_B}{U_B}$ and $U_E=\frac{U_A}{U_C}=\frac{U_A}{U_A\cap U_B}$. Then by second isomorphism theorem in group theory, there exists an isomorphism $G_0:U_F=\frac{U_AU_B}{U_B}\to\frac{U_A}{U_A\cap U_B}=U_E$ because U_B is a normal subgroup of U_G and U_A is a subgroup of U_G.

Then by 2(D)(t), $G:P(U_F)\to P(U_E)$ be defined by $G(A)=G_0(A)$ for all $A\in P(U_F)$, so $\mathcal{G}$ is an s-map of $\mathcal{F}$ to $\mathcal{E}$

iii) Now we show that $G_0|\bar{F}e:\bar{F}e\to\bar{E}e$ is an isomorphism for all $e\in F$. We have $\bar{F}e=\frac{\bar{D}eU_B}{U_B}=\frac{\bar{A}e\bar{B}e\ U_B}{U_B}$. Since $\bar{B}e$ is a subgroup of U_B, $\bar{B}eU_B=U_B$ and so $\bar{F}e=\frac{\bar{A}eU_B}{U_B}$, $\bar{E}e=\frac{\bar{A}eU_C}{U_C}=\frac{\bar{A}e\ U_A\cap U_B}{U_A\cap U_B}$.

Letting $G=U_G$, $C=\bar{A}e$, $B=U_B$, $A=U_A$ and $F=G_0$ in 2(A)(c) we get $G_0|\bar{F}e=G_0|\frac{\bar{A}eU_B}{U_B}:\frac{\bar{A}eU_B}{U_B}\to\frac{\bar{A}e\ U_A\cap U_B}{U_A\cap U_B}=\bar{E}e$ is an isomorphism, since U_B is a normal subgroup of U_G, U_A is a subgroup of U_G and $\bar{A}e$ is a subgroup of U_A.

iv) Lastly, we show that $\bar{E}g=G_0\bar{F}$. Since g be the identity map, it is enough to show that for all $e\in F$, $\bar{E}e=G_0\bar{F}e$, and again, by 2(A)(c) we have, $G_0\bar{F}e=G_0(\frac{\bar{A}eU_B}{U_B})=\frac{\bar{A}e\ U_A\cap U_B}{U_A\cap U_B}=\bar{E}e$. Therefore $G_0\bar{F}e=\bar{E}e$.

Third Isomorphism Theorem for s-Groups

Theorem 3.5 *For any s-group $\mathcal{G}$ and for any pair of s-normal subgroups $\mathcal{A},\mathcal{B}$ of $\mathcal{G}$ such that $\mathcal{A}\subseteq\mathcal{B}$, $\frac{\mathcal{G}}{\mathcal{B}}|A$ is purely s-isomorphic to $\frac{\frac{\mathcal{G}}{\mathcal{A}}}{\frac{\mathcal{B}}{\mathcal{A}}}$.*

Proof: Let $\frac{\mathcal{G}}{\mathcal{B}}|A=\mathcal{C}$. Then $C=B\cap A=A$, $U_C=\frac{U_G}{U_B}$ and $\bar{C}c=\frac{\bar{G}cU_B}{U_B}$ for all $c\in C$.

Let $\frac{\mathcal{G}}{\mathcal{A}}=\mathcal{D}$. Then $D=G\cap A=A$, $U_D=\frac{U_G}{U_A}$ and $\bar{D}d=\frac{\bar{G}dU_A}{U_A}$ for all $d\in D$.

Let $\frac{\mathcal{B}}{\mathcal{A}}=\mathcal{E}$.Then $E=B\cap A=A$, $U_E=\frac{U_B}{U_A}$ and $\bar{E}e=\frac{\bar{B}eU_A}{U_A}$ for all $e\in E$.

Let $\frac{\mathcal{D}}{\mathcal{E}}=\mathcal{F}$. Then $F=D\cap E=A\cap A=A$, $U_F=\frac{U_D}{U_E}$ and $\bar{F}f=\frac{\bar{D}fU_E}{U_E}$ for all $f\in F$.

Now we show that there is a pure s-isomorphism $\mathcal{H}=(h,H):\mathcal{C}\to\mathcal{F}$, where $h:C\to F$ and $H:P(U_C)\to P(U_F)$.

First observe that,

(i) $C=A=F$. Let h be the identity map of C onto F. Then h is a bijection.

(ii) $U_C=\frac{U_G}{U_B}$ and $U_F=\frac{U_D}{U_E}=\frac{\frac{U_G}{U_A}}{\frac{U_B}{U_A}}$. Then, by third isomorphism theorem in group theory, we have an isomorphism $H_0:U_C=\frac{U_G}{U_B}\to\frac{\frac{U_G}{U_A}}{\frac{U_B}{U_A}}=U_F$ because U_A,U_B are normal subgroups of U_G.

Then by 2(D)(t), $H:P(U_C)\to P(U_F)$ be defined by $H(A)=H_0(A)$ for all $A\in P(U_C)$, so $\mathcal{H}$ is an s-map of $\mathcal{C}$ to $\mathcal{F}$

(iii) Now we show that, $H_0|\bar{C}a: \bar{C}a \to \bar{F}a$ is an isomorphism for all $a \in C$. Since $\bar{C}a = \frac{\bar{G}aU_B}{U_B}$ and $\bar{F}a = \frac{\bar{D}aU_E}{U_E} = \frac{\frac{\bar{G}aU_AU_B}{U_A \quad U_A}}{\frac{U_B}{U_A}}$, letting $G = U_G$, $B = U_B$, $A = U_A$ and $C = \bar{G}a$ in 2(A)(d), $H_0|\bar{C}a = H_0|\frac{\bar{G}aU_B}{U_B} : \frac{\bar{G}aU_B}{U_B} \to \frac{\frac{\bar{G}aU_AU_B}{U_A \quad U_A}}{\frac{U_B}{U_A}}$ is an isomorphism, since U_A, U_B are normal subgroups of U_G such that $U_A \subseteq U_B$ and $\bar{G}a$ is any subgroup of U_G.

(iv) Lastly, we show that $\bar{F}h = H_0\bar{C}$. Since h is the identity map, it is enough to show that for all $a \in C$, $\bar{F}a = H_0\bar{C}a$. Again by 2(A)(d), $H_0\bar{C}a = H_0(\frac{\bar{G}aU_B}{U_B}) = \frac{\frac{\bar{G}aU_AU_B}{U_A \quad U_A}}{\frac{U_B}{U_A}} = \bar{F}a$. Therefore $H_0\bar{C}a = \bar{F}a$.

IV. CONCLUSION

In this paper we generalized the existing correspondence and isomorphism theorems of groups in the crisp setup to those of generalized soft group.

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