IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

Vol. 7, Issue 2, February 2020

More on the Free Factors of Parafree Lie Algebras

İbrahim Halil Coşkun

Harran University, Department of Mathematics, Şanlıurfa, Turkey

Abstract: Parafree Lie algebras are algebras that are residually nilpotent and satisfy the property that their quotients by the terms of the lower central series are isomorphic to the corresponding quotients of a free Lie algebra. There are many nonfree parafree Lie algebras and they all have many properties in common with free Lie algebras. In this paper, our main goal is to investigate the free product of parafree Lie algebras. In particular, we prove that a finite number of free factors of a parafree Lie algebra are parafree.

Keywords: Free Lie algebras, Parafree Lie algebras, Free Product, Free Factors

I. INTRODUCTION

Let L be a Lie algebra over a field k. The series

$$L=\gamma_1(L)\supset\gamma_2(L)\supset\gamma_3(L)\supset\ldots\supset\gamma_k(L)\supset\ldots$$

is called lower central series of L and defined

$$\gamma_1(L)=L,\ \gamma_2(L)=[L,L],\ \gamma_3(L)=[L,\gamma_2(L)],\ldots,\gamma_k(L)=[L,\gamma_{k-1}(L)],\ldots$$

If k is the smallest positive integer that satisfying $\gamma_k(L)=\{0\}$ then L is called nilpotent of class k. The quotient algebras $L/\gamma_2(L)$, $L/\gamma_3(L)$, $L/\gamma_4(L)$,…are called the lower central sequence of L.

Let F and L be two Lie algebras. For all k≥1, if we have

$$L/\gamma_k(L) \cong F/\gamma_k(F)$$

then we call that ‘’L and F have the same lower central sequence’’.

Let L be a Lie algebra over a field k. For any 0≠g∈L if there exists a homomorphism ψ_g form L to a nilpotent Lie algebra N such that $\psi_g(g)\neq 0$ then L is called residually nilpotent.
For more details about the Lie algebra theory, see [Erdmann and Wildon, 2006; Jacobson, 1979].

A Lie algebra L is called parafree if it is residually nilpotent and has the same lower central sequence as a free Lie algebra. Let $B\subset L, L\equiv \bar{L}(mod\ \gamma_2(L))$ and $\equiv \bar{B}(mod\ \gamma_2(L))$. If $\bar{B}$ freely generates $\bar{L}$, then it is called that ‘’ B freely generates L modulo $\gamma_2(L)$’’.

Parafree Lie algebras firstly have defined by Baur (1978).These special Lie algebras satisfy properties that are analogous to those of parafree groups (Baumslag, 1967, 1969). Velioğlu (2013) has proved that a subalgebra and the quotient algebra of a parafree Lie algebra are again parafree. Moreover Ekici and Velioğlu (2014) have studied on unions of parafree Lie algebras and they have proved that the direct limit of a system of parafree Lie algebras is parafree (2015). Velioğlu (2019) has obtained some results about parafree metabelian Lie algebras. Later Ekici and Velioğlu (2019) have investigated the metabelian product of parafree Lie algebras. Despite all these studies, the theory on parafree Lie algebras has still answered questions .We have taken this opportunity to obtain some results about parafree Lie algebras.
The aim of this work is to investigate the free product of parafree Lie algebras. In particular, we prove that a finite number of free factors of a parafree Lie algebra are parafree.

II. MAIN RESULTS

Definition 2.1. Let $(G_\alpha)_{\alpha\in I}$ be a family of Lie algebras over a field k with presentations (X_α/R_α) where for all $\alpha\neq\beta$ we have $X_\alpha\cap X_\beta=\emptyset$. Set $X=\bigcup_\alpha X_\alpha$, $R=\bigcup_\alpha R_\alpha$. Then $G=(X,R)$ is called the free product of family $(G_\alpha)_{\alpha\in I}$ and it is denoted by $G=\prod_\alpha * G_\alpha$.
There is a canonical epimorphism from the free product $\prod_\alpha * G_\alpha$ to the direct sum $\oplus_\alpha G_\alpha$ of the family $(G_\alpha)_{\alpha\in I}$. This epimorphism is defined by identity map of algebras $(G_\alpha)_{\alpha\in I}$ to theirself.
If $I=\{1,2,\ldots,n\}$, it is simply written

$$G=G_1*G_2*\ldots*G_n$$

DOI 10.17148/IARJSET.2020.7201

The algebras $G_1, G_2, \dots, G_n$ are called the free factors of G. Note that for any $\alpha \in I$ there exists a homomorphism $I_\alpha : G_\alpha \to G$, extending the identity map of X_α to X (for more details see [Bahturin, 1987]).

Baur 1978 studied on free product of parafree Lie algebras and he proved that the free product of a finite number of parafree Lie algebra is parafree by using homological methods.
Let $R = R_1 * R_2$ be a parafree Lie algebra of two Lie algebras R_1, R_2. Velioğlu 2013 has proved that the free factors R_1, R_2 of parafree Lie algebra R are parafree. We want to generalise this result by proving it for a finite numbers free factors of a parafree Lie algebra. We carry the formal arguments used in Velioğlu 2013. To do that we need the following propositions.

Proposition 2.1. Let $\mathrm{A}_1, \dots, A_n$ be a finite number of parafree Lie algebras. Then we have the following equation.

$$\mathrm{A}_1/\gamma_2(\mathrm{A}_1) \oplus \cdots \oplus \mathrm{A}_n/\gamma_2(A_n) = \mathrm{A}_1 \oplus \cdots \oplus \mathrm{A}_n / \gamma_2(\mathrm{A}_1) \oplus \cdots \oplus \gamma_2(A_n) \qquad (2.1)$$

Proof. Consider a element $(a_1 + \gamma_2(\mathrm{A}_1)) + \cdots + (a_n + \gamma_2(A_n))$ of the algebra $\mathrm{A}_1/\gamma_2(\mathrm{A}_1) \oplus \cdots \oplus \mathrm{A}_n/\gamma_2(\mathrm{A}_n)$.

It is clear that

$$(a_1 + \gamma_2(\mathrm{A}_1)) + \cdots + (a_n + \gamma_2(A_n))$$
$$=(a_1 + \cdots + a_n + \gamma_2(\mathrm{A}_1) + \cdots + \gamma_2(A_n)) \in \mathrm{A}_1 \oplus \cdots \oplus \mathrm{A}_n / \gamma_2(\mathrm{A}_1) \oplus \cdots \oplus \gamma_2(\mathrm{A}_n). \qquad (2.2)$$

On the other hand, for any element $(a_1 + \cdots + a_n + \gamma_2(\mathrm{A}_1) + \cdots + \gamma_2(A_n))$ of the algebra $\mathrm{A}_1 \oplus \cdots \oplus \mathrm{A}_n / \gamma_2(\mathrm{A}_1) \oplus \cdots \oplus \gamma_2(\mathrm{A}_n)$, we have

$$(a_1 + \cdots + a_n + \gamma_2(\mathrm{A}_1) + \cdots + \gamma_2(A_n))$$
$$= (a_1 + \gamma_2(\mathrm{A}_1) + \cdots + a_n + \gamma_2(A_n)) \in \mathrm{A}_1/\gamma_2(\mathrm{A}_1) \oplus \cdots \oplus \mathrm{A}_n/\gamma_2(A_n). \qquad (2.3)$$

Therefore by (2.2) and (2.3), we prove the equality (2.1).

Proposition 2.2. Let $\mathrm{A}_1, \dots, A_n$ be a finite number of parafree Lie algebras and $R = \mathrm{A}_1 * \cdots * A_n$ be the free product of algebras $\mathrm{A}_1, \dots, A_n$. Then

$$\mathrm{R}/\gamma_2(\mathrm{R}) = \mathrm{A}_1/\gamma_2(\mathrm{A}_1) \oplus \cdots \oplus \mathrm{A}_n/\gamma_2(A_n).$$

Proof. Let $u \in R$ and consider a homomorphism defined as

$$\sigma : \mathrm{R} \to \mathrm{A}_1 \oplus \cdots \oplus \mathrm{A}_n / \gamma_2(\mathrm{A}_1) \oplus \cdots \oplus \gamma_2(\mathrm{A}_n)$$

$$u + \gamma_2(\mathrm{R}) \to u + \gamma_2(\mathrm{A}_1) + \cdots + \gamma_2(\mathrm{A}_n).$$

By the definition of σ, it is clear that $\mathrm{Ker}\,\sigma = \gamma_2(\mathrm{R})$ and σ is a surjective homomorphism. By isomorphism theorems we have

$$\mathrm{R}/\gamma_2(\mathrm{R}) \cong \mathrm{A}_1 \oplus \cdots \oplus \mathrm{A}_n / \gamma_2(\mathrm{A}_1) \oplus \cdots \oplus \gamma_2(\mathrm{A}_n).$$

Therefore by the equality (2.1), we have

$$\mathrm{R}/\gamma_2(\mathrm{R}) \cong \mathrm{A}_1/\gamma_2(\mathrm{A}_1) \oplus \cdots \oplus \mathrm{A}_n/\gamma_2(A_n).$$

Lemma 2.1. Let $R = \mathrm{A}_1 * \cdots * A_n$ be a parafree Lie algebra. Then the free factors $\mathrm{A}_1, \cdots, A_n$ are parafree.

Proof. We want to prove that for $i = 1, \dots, n$ the algebras A_i are residually nilpotent and each one has the same lower central sequence as a free Lie algebra. Since R is parafree then it is residually nilpotent. Therefore for $i = 1, \dots, n$, algebras A_i are residually nilpotent [Velioğlu, 2013].
Now by the Proposition 2.2, we have

$$R/\gamma_2(R) \cong A_1/\gamma_2(A_1) \oplus \ldots \oplus A_n/\gamma_2(A_n).$$

For $i = 1, \ldots, n$, let sets X_i be generating sets of Lie algebras A_i such that X_i are linearly independent modulo $\gamma_2(A_i)$. Therefore by Proposition 2.2, the set $X_1 \cup \cdots \cup X_n$ is linearly independent modulo $\gamma_2(R)$ and it freely generates parafree Lie algebra R modulo $\gamma_2(R)$. Also we know that for $i = 1, \ldots, n$ the sets X_i freely generate Lie algebras A_i modulo $\gamma_2(A_i)$.

On the other hand quotient algebra $R/\gamma_n(R)$ is a free nilpotent Lie algebra that freely generated by the set $X_1 \cup \cdots \cup X_n$. Hence by Bahturin (1978), Theorem 9, for $i = 1, \ldots, n$, the sets X_i generate a free nilpotent Lie algebra modulo $\gamma_n(A_i)$. Therefore each Lie algebra A_i has the same lower central sequence as a free Lie algebra. Finally we have that the free factors A_i are parafree.

REFERENCES

[1]. G. Baumslag, ''Groups with the same lower central sequence as a relatively free group I. The groups'', Trans. Amer. Math. Soc., 129, 308-321, 1967.
[2]. G. Baumslag, ''Groups with the same lower central sequence as a relatively free group. II Properties'', Trans. Amer. Math. Soc., 142, 507-538, 1969.
[3]. H. Baur, ''Parafreie Lie algebren und Homologie'', Diss. Eth Nr., 6126, 1978.
[4]. K. Erdmann, MJ Wildon, ''Introduction to Lie algebras'', Springer, 2006.
[5]. N. Ekici, Z.Velioğlu, ''Unions of Parafree Lie Algebras'', Hindawi Publishing Corporation Algebra, Vol. 2014, ID: 385397,1-5, 2014.
[6]. N. Ekici, Z. Velioğlu, ''Direct Limit of Parafree Lie Algebras'', Journal of Lie Theory, 25(2), 477-484, 2015.
[7]. Y. Bahturin, ''Identical Relations in Lie Algebras'', VNU Science Press, Utrecht, 1987.
[8]. Z. Velioğlu,''Direct Sum and Free Factors of Parafree Lie Algebras, International Journal Pure & Applied Mathematics'',83(4),601-606, 2013.
[9]. Z. Velioğlu, ''Subalgebras and Quotient Algebras of Parafree Lie Algebras'', International Journal Pure and Applied Mathematics, 83(3),507-514, 2013.
[10]. Z.Velioğlu, N.Ekici, ''Paraserbest Lie Cebirlerinin Metabelyen Çarpımı, 2-Simetrik Kelimeler ve Verbal Alt Cebir'', Erzincan Üniversitesi Fen Bilimleri Dergisi, 12(2),771-777, 2019.
[11]. Z.Velioğlu, ''Parafree Metabelian Lie Algebras which are Determined by Parafree Lie Algebras'', Communications Faculty Sciences University of Ankara Series, A1:Mathematics and Statistics, 68(1),883-888, 2019.