IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

Vol. 7, Issue 7, July 2020

On a Class of Univalent Functions with Negative Coefficients defined by General Linear Operator

N.D. Sangle[1], P.S. Mane[2], P.T. Bathe[3]

Associate Professor, Department of Mathematics, Annasaheb Dange College of Engineering & Technology, Ashta, India[1]

Research Student, Department of Mathematics, Shivaji University, Kolhapur, india[2]

Assistant Professor, Department of Mathematics, Rajarshi Chhatrapati Shahu College, Kolhapur, India[3]

Abstract: In this study $S_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B})$ of an univalent function with negative coefficients which is defined by a new general linear operator $H_m^{s,c}$ have been introduced. The sharp results for coefficient estimators, distortion and closure bounds, Hadamard product and Neighbourhood, and this paper deals with the utilizing of many of the results for classical hypergeometric functions, where there can be generalized to m-hypergeometric functions .

Subclasses of univalent functions are presented, and it has involving operator $H_m^{s,c}(\mathrm{c}_i,\mathrm{b}_j)$ which generalizes many well –known. Denote A be the class function f and other results have been studied.

Keywords: Univalent functions, coefficient estimator, linear operator, neighbourhood

1 INTRODUCTION

Many researchers such as Mohammed and Darus[8], Adweby and Darus[1] and others have used the m-hypergeometric function for studying certain families of mathematic viable functions in an open unit disk. The m-hypergeometric functions are generalized configuration of the classical hypergeometric functions. Then by assuming the limit $m \to 1$, It would return to a classical hypergeometric functions. The formal set of hypergeometric functions have been used and introduced by many famous researchers were started by Euler in (1748), Guass (1813) and Cauchy (1852) see[4]. Also, it was converted a simple notation $\lim\limits_{m\to 1}\frac{1-m^c}{1-m}=c$ into a symmetric theory of hypergeometric function in same trend of theory of Gauss hypergeometric function.

Here this study deals with the utilizing of many of the results for classical hypergeometric function, where there can be generalized to m-hypergeometric functions.

In this work, a subclass of univalent function is introduced, and it has involving operator $H_m^{s,c}(\mathrm{c}_i,\mathrm{b}_j)$ which generalizes many well-known. Denote A be a class of functions f of the form

$$f(\mathrm{z})=\mathrm{z}+\sum_{n=2}^{\infty}a_n z^n \tag{1.1}$$

Which are analytic and univalent in the open unit disc $\boldsymbol{U}=\{\boldsymbol{z}\in\mathbf{C}:|\boldsymbol{z}|<\mathbf{1}\}$. A function $f\in A$ is said to be starlike of complex order if the following condition (see [2]) is satisfied.

$$Re\left\{\frac{\frac{\mathrm{z}(f'(\mathrm{z}))}{f(\mathrm{z})}-1}{2\delta\left(\frac{\mathrm{z}(f'(\mathrm{z}))}{f(\mathrm{z})}-\mu\right)-\left(\frac{\mathrm{z}(f'(\mathrm{z}))}{f(\mathrm{z})}-1\right)}\right\}>\beta,\left(0\le\mu<\frac{1}{2\delta},0<\beta\le 1,\frac{1}{2}\le\delta\le 1\right). \tag{1.2}$$

For complex parameters $c_1,c_2,...,\mathrm{c}_t$ and $b_1,b_2,...,b_r$ where $b_j\in\mathbf{C}\backslash\{0,-1,-2,\dots\}$, $(\mathrm{j}=1,2,...,\mathrm{r},|\,\mathrm{m}\,|<1)$, the m-hypergeometric

$$ {}_t\varphi_r = \sum_{n=0}^{\infty} \frac{(c_1, m)_n \dots (c_t, m)_n}{(m,m)_n (b_1, m)_n \dots (b_r, m)_n} z^n \tag{1.3}$$

$(t = r+1$ such that $t, r \in N_0 = \{0,1,2,3,\dots\}; z \in U)$. The m-shifted factorial is involving by $(c,m)_0 = 1$ and $(c,m)_n = (1-c)(1-cm)(1-cm^2)\dots(1-cm^{n-1}),\ n \in N,$ where c be any complex number and in terms of the Gamma function

$$(m^{\mu}, m)_n = \frac{\Gamma_m(\mu+n)(1-m)^n}{\Gamma_m(\mu)} \text{ such that } \Gamma_m(y) = \frac{(m,m)_\infty (1-m)(1-y)}{(m^y, m)_\infty}, 0 < m < 1.$$

The study suggests that by utilizing ratio test, the series (1.3) converges absolutely in open unit disc U , |m|<1

$$ {}_2\varphi_1 = \sum_{n=0}^{\infty} \frac{(c_1, m)_n (c_2, m)_n}{(m,m)_n (b_1, m)_n} z^n \qquad (|m| < 1, z \in U).$$

Is the m-hypergeometric function see [2], [3]. Recently Mohammed and Darus[8] defined the following : $I(c_i, b_j; m) f : A \to A$

$$I(c_i, b_j; m) f(z) = z + \sum_{n=2}^{\infty} \frac{(c_1, m)_{n-1} \dots (c_t, m)_{n-1}}{(m,m)_{n-1} (b_1, m)_{n-1} \dots (b_r, m)_{n-1}} a_n z^n.$$

The Srivastava-Attiya operator $T_{s,c} : A \to A$ defined in [9] as:

$$T_{s,c} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+c}{n+c} \right)^s a_n z^n, \tag{1.4}$$

where $z \in U, c \in \mathbb{C}\backslash\{0,-1,-2,\dots\}, s \in \mathbf{C}$ and $f \in A$. This linear operator $T_{s,c}$ can be written as

$$T_{s,c} f(z) = G_{s,c}(z) * f(z) = (1+c)^s \left(\varphi(z,s,c) - c^{-s} \right) * f(z),$$

by utilizing the Hadamard product (convolution). Here $\varphi(z,s,c) = \sum_{n=0}^{\infty} \frac{z^n}{(n+c)^s}$ is well known Hurwitz-Lerch zeta function (see [9], [10]), It is also an important function of Analytic Number Theory such the De-Jonquiere function:

$$H_{i,s}(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n)^s} = z\phi(z,s,1), \qquad (\mathrm{Re}(s) > 1 \text{ if } |z| = 1).$$

We can define the linear operator $H_m^{s,c}(c_i, b_j) f : A \to A$ as follow:

$$H_m^{s,c}(c_i, b_j) f(z) = z + \sum_{n=2}^{\infty} \frac{(c_1, m)_{n-1} \dots (c_t, m)_{n-1}}{(m,m)_{n-1} (b_1, m)_{n-1} \dots (b_r, m)_{n-1}} \left(\frac{1+c}{n+c} \right)^s a_n z^n \tag{1.5}$$

$z \in U, c \in \mathbb{C}\backslash\{0,-1,-2,\dots\}, s \in \mathbf{C}, c_i, b_j \in \mathbf{C}\backslash\{0,-1,-2,\dots\}, |m| < 1$ and $t = r+1$. It should be noted that the linear operator (1.5) introduced by A.R.S. Juma and M.Darus [4].

Definition 1.1. *Let f be a function and $f \in U$ is said to be in the class $R_m^{s,c}(\mu, \beta, \delta, A, B)$ if the following condition holds:*

$$\left| \frac{\dfrac{z[H_m^{s,c}(c_i, b_j) f(z)]'}{H_m^{s,c}(c_i, b_j) f(z)} - 1}{(A-B)\delta \left(\dfrac{z[H_m^{s,c}(c_i, b_j) f(z)]'}{H_m^{s,c}(c_i, b_j) f(z)} - \mu \right) + B \left(\dfrac{z[H_m^{s,c}(c_i, b_j) f(z)]'}{H_m^{s,c}(c_i, b_j) f(z)} - 1 \right)} \right| < \beta, \tag{1.6}$$

where $0 \le \mu < \frac{1}{2\delta}, 0 < \beta \le 1, \frac{1}{2} \le \delta \le 1, -1 \le B < A < 1$ and $z \in U$.

Let T denote the subclass of A consist of function of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geq 0 \tag{1.7}$$

Now we define the class $S_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B})$ by

$$S_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B}) = R_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B}) \bigcap \mathrm{T}.$$

The study we have the following class and confirm that by specializing the parameters $\mu,\beta,\delta,\mathrm{A},\mathrm{B}$

1. The class $S_m^{-k,0}(\alpha,\beta,\xi,1,-1)$ is the class studied by A.R.S.Juma and S.R. Kulkarni [3].
2. The class $S_m^{-k,0}(0,1,1,1,-1)$ is precisely the class of starlike function in U.
3. The class $S_m^{-k,0}(\mu,1,1,1,-1)$ is the class of starlike function of order $\mu(0 \leq \mu < 1)$.
4. The class $S_m^{-k,0}\left(0,\beta,\frac{\mu+1}{2},1,-1\right)$ is the class studied by Lakshminar-Simhan [7].
5. The class $S_m^{-k,c}\left(\mu,\beta,\delta,1,-1\right)$ is the class studied by S.R.Kulkarni [6].

2 Coefficient estimates and other properties

Theorem 2.1: *Let f be defined by (1.7), then $f \in S_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B})$ if and only if*

$$\sum_{n=2}^{\infty} \frac{(\mathrm{c}_1,\mathrm{m})_{n-1}\ldots(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}\ldots(\mathrm{b}_r,\mathrm{m})_{n-1}} \left(\frac{1+c}{n+c}\right)^s [(\mathrm{n}-1)(1+\mathrm{B}\beta) - (\mathrm{A}-\mathrm{B})\beta\delta(\mathrm{n}-\mu)] a_n \leq (\mathrm{A}-\mathrm{B})\beta\delta(1-\mu) \tag{2.1}$$

$$0 \leq \mu < \frac{1}{2\delta}, 0 < \beta \leq 1, \frac{1}{2} \leq \delta \leq 1, -1 \leq B < A < 1.$$

Proof: If |z|=1, then

$$\left| z\left(H_m^{s,c}(\mathrm{c}_i,\mathrm{b}_j)f(\mathrm{z})\right)' - H_m^{s,c}(\mathrm{c}_i,\mathrm{b}_j)f(\mathrm{z}) \right| - \beta \left| (\mathrm{A}-\mathrm{B})\delta\left[z\left(H_m^{s,c}(\mathrm{c}_i,\mathrm{b}_j)f(\mathrm{z})\right)' - \mu H_m^{s,c}(\mathrm{c}_i,\mathrm{b}_j)f(\mathrm{z}) \right] + \beta[\mathrm{z}\left(H_m^{s,c}(\mathrm{c}_i,\mathrm{b}_j)f(\mathrm{z})\right)' - H_m^{s,c}(\mathrm{c}_i,\mathrm{b}_j)f(\mathrm{z})] \right|.$$

By utilizing (1.5)

$$= \left| \sum_{n=2}^{\infty} \frac{(\mathrm{c}_1,\mathrm{m})_{n-1}\ldots(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}\ldots(\mathrm{b}_r,\mathrm{m})_{n-1}} \left(\frac{1+c}{n+c}\right)^s a_n z^n - z - \frac{(\mathrm{c}_1,\mathrm{m})_{n-1}\ldots(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}\ldots(\mathrm{b}_r,\mathrm{m})_{n-1}} \left(\frac{1+c}{n+c}\right)^s \right|$$
$$-\beta \left| (\mathrm{A}-\mathrm{B})\delta \left[z + \sum_{n=2}^{\infty} \frac{(\mathrm{c}_1,\mathrm{m})_{n-1}\ldots(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}\ldots(\mathrm{b}_r,\mathrm{m})_{n-1}} \left(\frac{1+c}{n+c}\right)^s n a_n z^n - z\mu - \mu \sum_{n=2}^{\infty} \frac{(\mathrm{c}_1,\mathrm{m})_{n-1}\ldots(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}\ldots(\mathrm{b}_r,\mathrm{m})_{n-1}} \left(\frac{1+c}{n+c}\right)^s a_n z^n \right] \right.$$
$$\left. +\beta \left[z + \sum_{n=2}^{\infty} \frac{(\mathrm{c}_1,\mathrm{m})_{n-1}\ldots(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}\ldots(\mathrm{b}_r,\mathrm{m})_{n-1}} \left(\frac{1+c}{n+c}\right)^s n a_n z^n - z - \sum_{n=2}^{\infty} \frac{(\mathrm{c}_1,\mathrm{m})_{n-1}\ldots(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}\ldots(\mathrm{b}_r,\mathrm{m})_{n-1}} \left(\frac{1+c}{n+c}\right)^s a_n z^n \right] \right|$$

$$= \left| \sum_{n=2}^{\infty} (\mathrm{n}-1) \frac{(\mathrm{c}_1,\mathrm{m})_{n-1}\ldots(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}\ldots(\mathrm{b}_r,\mathrm{m})_{n-1}} \left(\frac{1+c}{n+c}\right)^s a_n z^n \right|$$
$$-\beta \left| (\mathrm{A}-\mathrm{B})\delta \left[(1-\mu)\mathrm{z} + \sum_{n=2}^{\infty} (\mathrm{n}-\mu) \frac{(\mathrm{c}_1,\mathrm{m})_{n-1}\ldots(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}\ldots(\mathrm{b}_r,\mathrm{m})_{n-1}} \left(\frac{1+c}{n+c}\right)^s a_n z^n \right] \right.$$
$$\left. +\beta \left[\sum_{n=2}^{\infty} (\mathrm{n}-1) \frac{(\mathrm{c}_1,\mathrm{m})_{n-1}\ldots(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}\ldots(\mathrm{b}_r,\mathrm{m})_{n-1}} \left(\frac{1+c}{n+c}\right)^s a_n z^n \right] \right|.$$

$$\le \sum_{n=2}^{\infty}\left[(n-1)+B\beta(n-1)+(A-B)\beta\delta(n-\mu)\right]\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s a_n z^n-(B-A)\beta\delta(1-\mu)\le 0$$

$$\sum_{n=2}^{\infty}\left[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu)\right]\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s a_n \le (B-A)\beta\delta(1-\mu).$$

By hypothesis thus by maximum modulus theorem, we get $f\in S_m^{s,c}(\mu,\beta,\delta,A,B)$ and versa suppose that $f\in S_m^{s,c}(\mu,\beta,\delta,A,B)$, therefore the condition (1.7) gives us

$$\left|\frac{\dfrac{z[H_m^{s,c}(c_i,b_j)f(z)]'}{H_m^{s,c}(c_i,b_j)f(z)}-1}{(A-B)\delta\left(\dfrac{z[H_m^{s,c}(c_i,b_j)f(z)]'}{H_m^{s,c}(c_i,b_j)f(z)}-\mu\right)+B\left(\dfrac{z[H_m^{s,c}(c_i,b_j)f(z)]'}{H_m^{s,c}(c_i,b_j)f(z)}-1\right)}\right|<\beta$$

$$\frac{\left|\sum_{n=2}^{\infty}\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s(n-1)a_n z^n\right|}{\left|(A-B)\delta\left[(1-\mu)+\sum_{n=2}^{\infty}(n-\mu)\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s a_n z^{n-1}\right]+\beta\left[\sum_{n=2}^{\infty}\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s(n-1)a_n z^{n-1}\right]\right|}<\beta$$

letting $z\to 1^-$ through real values. Then we get (2.1) the result is sharp for the function

$$f(z)=z-\frac{(A-B)\beta\delta(1-\mu)}{\left[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu)\right]\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s}a_n z^n, \quad \text{where } n\ge 2$$

Corollary 2.1. *Let* f *belonging to the class* $S_m^{s,c}(\mu,\beta,\delta,A,B)$. *Then*

$$a_n\le\frac{(A-B)\beta\delta(1-\mu)}{\left[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu)\right]\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s}, \tag{2.2}$$

where $n\ge 2$

Theorem 2.2. *Let* $f\in S_m^{s,c}(\mu,\beta,\delta,A,B)$. *Then for* $|z|\le r<1$, *we get*

$$r-r^2\frac{(A-B)\beta\delta(1-\mu)}{\frac{(c_1,m)...(c_t,m)}{(m,m)(b_1,m)\ ...(b_r,m)}\left(\frac{1+c}{2+c}\right)^s\left[(1+B\beta)+(A-B)\beta\delta(2-\mu)\right]}\le\left|H_m^{s,c}(c_i,b_j)f(z)\right|$$

$$\le r+r^2\frac{(A-B)\beta\delta(1-\mu)}{\frac{(c_1,m)...(c_t,m)}{(m,m)(b_1,m)...(b_r,m)}\left(\frac{1+c}{n+c}\right)^s\left[(1+B\beta)+(A-B)\beta\delta(2-\mu)\right]}$$

$$r-2r\frac{(A-B)\beta\delta(1-\mu)}{\frac{(c_1,m)...(c_t,m)}{(m,m)(b_1,m)...(b_r,m)}\left(\frac{1+c}{n+c}\right)^s\left[(1+B\beta)+(A-B)\beta\delta(2-\mu)\right]}\le\left|\left(H_m^{s,c}(c_i,b_j)f(z)\right)'\right|$$

$$\le 1+2r\frac{(A-B)\beta\delta(1-\mu)}{\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s\left[(1+B\beta)+(A-B)\beta\delta(2-\mu)\right]}$$

The above bounds are sharp.

Proof: By theorem (2.1) we have

IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

Vol. 7, Issue 7, July 2020

$$\sum_{n=2}^{\infty}[(\mathrm{n}-1)(1+\mathrm{B}\beta)-(\mathrm{A}-\mathrm{B})\beta\delta(\mu-n)]\frac{(\mathrm{c}_1,\mathrm{m})_{n-1}...(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}...(\mathrm{b}_r,\mathrm{m})_{n-1}}\left(\frac{1+c}{n+c}\right)^s a_n z^n \le (\mathrm{B}-\mathrm{A})\beta\delta(1-\mu),$$

then we have

$$\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s[(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)]a_n$$

$$\le \sum_{n=2}^{\infty}\frac{(\mathrm{c}_1,\mathrm{m})_{n-1}...(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}...(\mathrm{b}_r,\mathrm{m})_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(1+\mathrm{B}\beta)(\mathrm{n}-1)-(\mathrm{A}-\mathrm{B})\beta\delta(\mu-\mathrm{n})]a_n \le (\mathrm{A}-\mathrm{B})\beta\delta(1-\mu).$$

Then

$$\sum_{n=2}^{\infty}a_n \le \frac{(\mathrm{A}-\mathrm{B})\beta\delta(1-\mu)}{\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{n+c}\right)^s[(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)]}.$$

Hence

$$\left\|\left[H_m^{s,c}(\mathrm{c}_i,\mathrm{b}_j)\,f(z)\right]\right\| \le |\mathrm{z}|+|\mathrm{z}|^2\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s\sum_{n=2}^{\infty}a_n$$

$$\le r+r^2\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s\sum_{n=2}^{\infty}a_n \le r+\frac{r^2(\mathrm{A}-\mathrm{B})\beta\delta(1-\mu)}{(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)},$$

and

$$\left\|\left[H_m^{s,c}(\mathrm{c}_i,\mathrm{b}_j)\,f(z)\right]\right\| \ge |\mathrm{z}|-|\mathrm{z}|^2\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s\sum_{n=2}^{\infty}a_n$$

$$\ge r-r^2\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s\sum_{n=2}^{\infty}a_n \ge r-\frac{r^2(\mathrm{A}-\mathrm{B})\beta\delta(1-\mu)}{(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)}$$

thus (2.2) is true, further

$$\left|\left[H_m^{s,c}(\mathrm{c}_i,\mathrm{b}_j)\,f(z)\right]'\right| \le 1+2r\left[\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s\right]\sum_{n=2}^{\infty}a_n \le 1+\frac{2r(\mathrm{A}-\mathrm{B})\beta\delta(1-\mu)}{(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)}$$

And also

$$\left|\left[H_m^{s,c}(\mathrm{c}_i,\mathrm{b}_j)\,f(z)\right]'\right| \ge 1-\frac{2r(\mathrm{A}-\mathrm{B})\beta\delta(1-\mu)}{(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)}$$

The result is sharp for the function $f(\mathrm{z})$ defined by

$$f(\mathrm{z})=\mathrm{z}+\frac{(\mathrm{A}-\mathrm{B})\beta\delta(1-\mu)}{(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)}z^2$$

Theorem 2.3: $0<\beta\le 1, 0<\mu_1\le\mu_2<\frac{1}{2\delta}$ *and* $\frac{1}{2}\le\delta\le 1, -1\le\beta<A<1$ *then* $S_m^{s,c}(\mu_2,\beta,\delta,\mathrm{A},\mathrm{B})\subset S_m^{s,c}(\mu_1,\beta,\delta,\mathrm{A},\mathrm{B})$.

Proof: By utilizing assumption we get

$$\frac{(\mathrm{A-B})\beta\delta(1-\mu_2)}{\frac{(c_1,m)...(c_t,m)}{(m,m)(b_1,m)...(b_r,m)}\left(\frac{1+c}{2+c}\right)^s[(n-1)(1+\mathrm{B}\beta)-(\mathrm{A-B})\beta\delta(\mu_2-n)]}\le\frac{(\mathrm{A-B})\beta\delta(1-\mu_1)}{\frac{(c_1,m)...(c_t,m)}{(m,m)(b_1,m)...(b_r,m)}\left(\frac{1+c}{2+c}\right)^s[(n-1)(1+\mathrm{B}\beta)-(\mathrm{A-B})\beta\delta(\mu_1-n)]}$$

Thus $f\in S_m^{s,c}(\mu_1,\beta,\delta,\mathrm{A},\mathrm{B})$ implies that

$$\sum_{n=2}^{\infty}\frac{(\mathrm{A-B})\beta\delta(1-\mu_1)}{\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s}a_n\le\frac{(\mathrm{A-B})\beta\delta(1-\mu_2)}{\left[(n-1)(1+\mathrm{B}\beta)+(\mathrm{A-B})\beta\delta(n-\mu_2)\right]}$$

$$\le\frac{(\mathrm{A-B})\beta\delta(1-\mu_1)}{\left[(n-1)(1+\mathrm{B}\beta)+(\mathrm{A-B})\beta\delta(n-\mu_1)\right]}$$

Then $f\in S_m^{s,c}(\mu_1,\beta,\delta,\mathrm{A},\mathrm{B})$.

Theorem 2.4: *The set* $f\in S_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B})$ *is the convex set.*

Proof: Let $f_i(z)=z+\sum_{n=2}^{\infty}a_{n,i}z^n$, $(i=1,2)$ belongs to $S_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B})$ and let $g(z)=\delta_1F_1(z)+\delta_2F_2(z)$ with δ_1 and δ_2 non-negative and $\delta_1+\delta_2=1$ and we write

$$g(z)=z-\sum_{n=2}^{\infty}(\delta_1a_{n,1}+\delta_2a_{n,2})z^n.$$

It is suffices to show that $g(z)\in S_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B})$ that mean

$$\sum_{n=2}^{\infty}\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(1+\mathrm{B}\beta)(n-1)+(\mathrm{A-B})\beta\delta(n-\mu)](\delta_1a_{n,1}+\delta_2a_{n,2})$$

$$=\delta_1\sum_{n=2}^{\infty}\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(1+\mathrm{B}\beta)(n-1)+(\mathrm{A-B})\beta\delta(n-\mu)]\left[a_{n,1}\right]$$

$$+\delta_2\sum_{n=2}^{\infty}\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(1+\mathrm{B}\beta)(n-1)+(\mathrm{A-B})\beta\delta(n-\mu)]\left[a_{n,2}\right]$$

$$\le\delta_2(\mathrm{A-B})\beta\delta(1-\mu)+\delta_1(\mathrm{A-B})\beta\delta(1-\mu)=(\delta_1+\delta_2)(\mathrm{A-B})\beta\delta(1-\mu)$$

$$\le(\mathrm{A-B})\beta\delta(1-\mu).$$

Thus $g(z)\in S_m^{s,c}(\mu_1,\beta,\delta,\mathrm{A},\mathrm{B})$.

Thus study shall further try to obtain the extreme point in the following theorem.

Theorem 2.5. *Let* $f_1(z)=z$ and $f_n(z)$ *defined by*

$$f_n(z)=z+\frac{(\mathrm{A-B})\beta\delta(1-\mu)}{[(n-1)(1+\mathrm{B}\beta)+(\mathrm{A-B})\beta\delta(n-\mu)]\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s}z^n.$$

for all $n=2,3,...;0<\beta\le1,0\le\mu<\frac{1}{2\delta},\frac{1}{2}\le\delta\le1,-1\le B<A<1.$

Then $f(z)$ *is in the class* $S_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B})$ *if and only if it can be expressed in the form*

$$f(z)=\sum_{n=1}^{\infty}\gamma_nz^n \text{ where } \gamma_n\ge0 \text{ and } \sum_{n=1}^{\infty}\gamma_n=1 \text{ or } 1=\gamma_1+\sum_{n=2}^{\infty}\gamma_n.$$

IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

Vol. 7, Issue 7, July 2020

Proof: Let $f(z)=\sum_{n=1}^{\infty}\gamma_n z^n$ where $\gamma_n \geq 0$ *and* $\sum_{n=1}^{\infty}\gamma_n = 1$.

$$f(z)=z+\sum_{n=1}^{\infty}\frac{(A-B)\beta\delta(1-\mu)}{[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu)]\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s}\gamma_n z^n.$$

and we obtain

$$\sum_{n=1}^{\infty}\frac{[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu)]\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{2+c}\right)^s}{(A-B)\beta\delta(1-\mu)}\times\gamma_n\left[\frac{(A-B)\beta\delta(1-\mu)}{[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu)]\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s}\right]$$

$$=\sum_{n=1}^{\infty}\gamma_n = 1-\gamma_1 \leq 1.$$

In view of theorem (2.1), this shows that $f(z)\in S_m^{s,c}(\mu,\beta,\delta,A,B)$.

Conversely,

$$a_n \leq \frac{(A-B)\beta\delta(1-\mu)}{[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu)]\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s},\ n\geq 2.$$

If

$$\gamma_n=\frac{[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu)]\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s}{(A-B)\beta\delta(1-\mu)}$$

and $\gamma_1 = 1-\sum_{n=1}^{\infty}\gamma_n$, then we get

$$f(z)=\gamma_1 f_1(z)+\sum_{n=1}^{\infty}\gamma_n f_n(z).$$

3 Neighbourhood and Hadamard Product properties

Definition: 3.1. *Let* $\gamma \geq 0$, $f(z)\in T$ *on the (1.7) the* $(k,\gamma)-$*neighbourhood of a function* $f(z)$ *defined by*

$$N_{n,\gamma}(f)=\left\{g\in T: g(z)=z-\sum_{n=2}^{\infty}b_n z^n \text{ and } \sum_{n=2}^{\infty}n|a_n-b_n|\leq\gamma\right\}. \quad (3.1)$$

For the identity function $e(z)=z$, *we get*

$$N_{n,\gamma}(e)=\left\{g\in T: g(z)=z-\sum_{n=2}^{\infty}b_n z^n \text{ and } \sum_{n=2}^{\infty}n|b_n|\leq\gamma\right\}. \quad (3.2)$$

Theorem 3.1. *Let* $\gamma=\dfrac{2(A-B)\beta\delta(1-\mu)}{[(1+B\beta)+(A-B)\beta\delta(2-\mu)]\frac{(c_1,m)\ldots(c_t,m)}{(m,m)(b_1,m)\ldots(b_r,m)}\left(\frac{1+c}{2+c}\right)^s}$.

Then $S_m^{s,c}(\mu,\beta,\delta,A,B)\subset N_{n,\gamma}(e)$.

IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

Vol. 7, Issue 7, July 2020

Proof: Let $f \in S_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B})$. Then we get

$$[(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)]\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s\sum_{n=2}^{\infty}a_n$$

$$\le\sum_{n=2}^{\infty}[(\mathrm{n}-1)(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(\mathrm{n}-\mu)]\frac{(\mathrm{c}_1,\mathrm{m})_{n-1}...(\mathrm{c}_t,\mathrm{m})_{n-1}}{(\mathrm{m},\mathrm{m})_{n-1}(\mathrm{b}_1,\mathrm{m})_{n-1}...(\mathrm{b}_r,\mathrm{m})_{n-1}}\left(\frac{1+c}{n+c}\right)^s$$

$$\le(\mathrm{A}-\mathrm{B})\beta\delta(1-\mu),$$

therefore,

$$\sum_{n=2}^{\infty}a_n\le\frac{(\mathrm{A}-\mathrm{B})\beta\delta(1-\mu)}{[(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)]\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s}, \tag{3.3}$$

also we get $|\mathrm{z}|<r$

$$|f'(\mathrm{z})|\le 1+|\mathrm{z}|\sum_{n=2}^{\infty}na_n\le 1+r\sum_{n=2}^{\infty}na_n.$$

In the view of (3.3) we get

$$|f'(\mathrm{z})|\le 1+r\frac{2(\mathrm{A}-\mathrm{B})\beta\delta(1-\mu)}{[(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)]\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s}.$$

From the above inequality we have

$$\sum_{n=2}^{\infty}na_n\le\frac{2(\mathrm{A}-\mathrm{B})\beta\delta(1-\mu)}{[(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)]\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s}=\gamma,$$

thus $f\in N_{n,\gamma}(\mathrm{e})$.

Definition 3.2. *The function* $f(\mathrm{z})$ *defined by (1.7) is said to be a member of the class* $S_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B})$ *.if there exist a function* $g\in S_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B})$ *such that* $\left|\frac{f(\mathrm{z})}{g(\mathrm{z})}-1\right|\le 1-\varsigma$, $z\in U, 0\le\zeta<1$.

Theorem 3.2. *Let* $g\in S_m^{s,c}(\mu,\beta,\delta,\mathrm{A},\mathrm{B})$ *and* $\zeta=1-\frac{\gamma}{2}d(\mu,\beta,\delta,\mathrm{A},\mathrm{B})$. *Then* $N_{n,\gamma}(\mathrm{g})\subset S_m^{s,c}(\mu_1,\beta,\delta,\mathrm{A},\mathrm{B})$ *when* $0<\beta\le 1, 0\le\mu<\frac{1}{2\delta},\frac{1}{2}<\delta\le 1,-1\le B<A<1$ and $0\le\zeta<1$. *and*

$$d(\mu,\beta,\delta,\mathrm{A},\mathrm{B})=\frac{\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s\left[(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)\right]}{\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s\left[(1+\mathrm{B}\beta)(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)-(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)\right]}.$$

Proof: Let $F\in N_{n,\gamma}(\mathrm{g})$. Then by (3.3) we get $\sum_{n=2}^{\infty}n|a_n-\mathrm{b}_n|\le\gamma$ then $\sum_{n=2}^{\infty}|a_n-\mathrm{b}_n|\le\frac{\gamma}{2}$.

$$\sum_{n=2}^{\infty}b_n\le\frac{(\mathrm{A}-\mathrm{B})\beta\delta(1-\mu)}{[(1+\mathrm{B}\beta)+(\mathrm{A}-\mathrm{B})\beta\delta(2-\mu)]\frac{(\mathrm{c}_1,\mathrm{m})...(\mathrm{c}_t,\mathrm{m})}{(\mathrm{m},\mathrm{m})(\mathrm{b}_1,\mathrm{m})...(\mathrm{b}_r,\mathrm{m})}\left(\frac{1+c}{2+c}\right)^s},$$

therefore,

$$\left|\frac{F(z)}{g(z)}-1\right| \leq \frac{\sum_{n=2}^{\infty}|a_n-b_n|}{1-\sum_{n=2}^{\infty}b_n}$$

$$\leq \frac{\gamma}{2}\left[\frac{\frac{(c_1,m)...(c_t,m)}{(m,m)(b_1,m)...(b_r,m)}\left(\frac{1+c}{2+c}\right)^s[(1+B\beta)+(A-B)\beta\delta(2-\mu)]}{\frac{(c_1,m)...(c_t,m)}{(m,m)(b_1,m)...(b_r,m)}\left(\frac{1+c}{2+c}\right)^s[(1+B\beta)+(A-B)\beta\delta(2-\mu)-(A-B)\beta\delta(2-\mu)]}\right]$$

$$=\frac{\gamma}{2}(\mu,\beta,\delta,A,B)=1-\zeta.$$

Then by definition (3.2) we have $f \in S_m^{s,c}(\mu,\beta,\delta,A,B)$.

Theorem 3.3. *Let* $f(z)$ and $g(z) \in S_m^{s,c}(\mu,\beta,\delta,A,B)$ *be of the form such that*

$f(z)=z-\sum_{n=2}^{\infty}a_n z^n$ *and* $g(z)=z-\sum_{n=2}^{\infty}b_n z^n$ *when* $a_n \geq 0, b_n \geq 0$. *Then, the Hadamard product* $h(z)$ *defined by*

$h(z)=z-\sum_{n=2}^{\infty}a_n b_n z^n$ *is in the subclass* $S_m^{s,c}(\mu,\beta,\delta,A,B)$ *when*

$$\mu_2 \leq \frac{[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_1)]^2\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s-(A-B)\beta\delta(1-\mu_1)^2(n-1)(1+B\beta)-[(A-B)\beta\delta]^2(1-\mu_1)^2 n}{[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_1)]^2\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s-[(A-B)\beta\delta]^2(1-\mu_1)^2}.$$

Proof: By theorem (2.1) we get

$$\sum_{n=2}^{\infty}\frac{\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_1)]}{(A-B)\beta\delta(1-\mu_1)}a_n \leq 1 \tag{3.5}$$

And

$$\sum_{n=2}^{\infty}\frac{\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_1)]}{(A-B)\beta\delta(1-\mu_1)}b_n \leq 1 \tag{3.6}$$

We get only to find the largest μ_2 such that

$$\sum_{n=2}^{\infty}\frac{\frac{(c_1,m)_{n-1}...(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}...(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_2)]}{(A-B)\beta\delta(1-\mu_2)}a_n b_n \leq 1.$$

Now by Cauchy-Schwartz inequality we get,

IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

Vol. 7, Issue 7, July 2020

$$\sum_{n=2}^{\infty}\frac{\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_2)]}{(A-B)\beta\delta(1-\mu_2)}\sqrt{a_nb_n}\le 1.$$

We only to show that

$$\frac{\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_2)]}{(A-B)\beta\delta(1-\mu_2)}a_nb_n$$

$$\le\frac{\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_1)]}{(A-B)\beta\delta(1-\mu_1)}\sqrt{a_nb_n}$$

equivalently

$$\frac{(A-B)\beta\delta(1-\mu_2)}{\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_2)]}\times\frac{\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_1)]}{(A-B)\beta\delta(1-\mu_1)}$$

But from (3.7) we get

$$\sqrt{a_nb_n}\le\frac{(A-B)\beta\delta(1-\mu_1)}{\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_1)]}$$

Consequently, we also need to p rove that

$$\frac{(A-B)\beta\delta(1-\mu_1)}{\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_1)]}\le\frac{(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_1)}{(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_2)}$$

or equivalently, that

$$\mu_2\le\frac{[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_1)]^2\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s-(A-B)\beta\delta(1-\mu_1)^2(n-1)(1+B\beta)-[(A-B)\beta\delta]^2(1-\mu_1)^2n}{[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu_1)]^2\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s-[(A-B)\beta\delta]^2(1-\mu_1)^2}.$$

Theorem3.4. *Let* $f(z)$ and $g(z)\in S_m^{s,c}(\mu,\beta,\delta,A,B)$ *be defined by (1.7) and* $q>-1$. *Then the function* $G(z)$ *defined as* $G(z)=\frac{q+1}{z^q}\int_0^z w^{q-1}f(w),\quad q>-1,$ *also belongs to* $S_m^{s,c}(\mu,\beta,\delta,A,B)$.

Proof: By virtue of $G(z)$ it is follows form (1.7) that

$$G(z)=\frac{q+1}{z^q}\int_0^w\left[w^q-\sum_{n=2}^{\infty}a_nw^{n+q-1}\right]dw$$

$$=z-\sum_{n=2}^{\infty}\left(\frac{q+1}{q+n}\right)a_nz^n.$$

But

$$\sum_{n=2}^{\infty}\frac{\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu)]}{(A-B)\beta\delta(1-\mu)}\left(\frac{q+1}{q+n}\right)a_n\le 1.$$

Since $\frac{q+1}{q+n}\le 1$ and by theorem (2.1), so the proof is complete.

Theorem 3.5. *Let* $F(z)\in S_m^{s,c}(\mu,\beta,\delta,A,B)$ *be defined by (1.7) and*

$$f_\alpha(z)=(1-\alpha)z+\alpha\int_0^z\frac{f(w)}{w}dw,\qquad(\alpha\ge 0,\ z\in U).$$

Then $f_\alpha(z)$ *is also in* $S_m^{s,c}(\mu,\beta,\delta,A,B)$ *if* $0\le\alpha\le 2$.

Proof: Let f be defined by (1.7) then

$$f_\alpha(z)=(1-\alpha)z+\alpha\int_0^z\frac{w-\sum_{n=2}^{\infty}a_nw^n}{w}dw$$

$$=z-\sum_{n=2}^{\infty}\frac{aa_nz^n}{n}.$$

By theorem (2.1) since $\left(\frac{\alpha}{2}\le 1\right)$ we get,

$$\sum_{n=2}^{\infty}\frac{\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu)]}{(A-B)\beta\delta(1-\mu)}\left(\frac{\delta}{n}\right)a_n$$

$$\le\frac{\frac{(c_1,m)_{n-1}\ldots(c_t,m)_{n-1}}{(m,m)_{n-1}(b_1,m)_{n-1}\ldots(b_r,m)_{n-1}}\left(\frac{1+c}{n+c}\right)^s[(n-1)(1+B\beta)+(A-B)\beta\delta(n-\mu)]}{(A-B)\beta\delta(1-\mu)}\left(\frac{\delta}{n}\right)a_n\le 1.$$

Then $f_\alpha(z)$ is in $S_m^{s,c}(\mu,\beta,\delta,A,B)$.

REFERENCES

[1] Aldweby H. Darus M. Univalence of a new general integral operator associated with the q-hypergeometric function Inter. Jour. *Math.Sci.*,ID769537, p.

[2]Carlson B.C., er. D. B. (1984). Starlike and presfar like hypergeometric functions. SIAM J.Math. Anal., vol.15, pp. 737-745.

[3]Gasper G.,Rahman M. (1990)Basic Hypergeometric Series. Encyclopedia of Mathematics and its Applications, Cambridge University Press, Vol.18, pp.207-216

[4]Juma. A. R. S. and M.Darus (2018). Certain subclasses of analytic functions defined by a new general linear operator, *Matematka*.vol, 24.,PP.24- 36.

[5]Juma. A. R. S. and S. R. Kulkarni.(2007). On univalent function with negative coefficients by using generalized Salagean operator. Filomat,pp173- 184.

[6]Kulkarni, S.R.,(1981). Some problems connected with univalent functions, Ph.D. Thesis, Shivaji University, Kolhapur (unpublished)).

[7]Lakshminar - simhan, T.V., (1977). On subclasses of function starlike in the unit disc, J.*Indian Math.* Soc., 41, 233-243.

[8]Mohammed A., Darus M. (2013). A generalized operator involving the q-hypergeometric function. *MatematickziVesn'ik*, vol.65, no.4, pp. 454- 465.

IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

Vol. 7, Issue 7, July 2020

[9]Srivastava H.M., Attiya A.A. (2007). An integral operator associated with the Hurwitz Lerchzet function and differential subordination. Integral transforms Spec. Funct., Vol.18, pp. 207-216.

[10]Srivastava H.M., Attiya A.A. (2007). An integral operator associated with the Hurwitz Lerchzet function and differential subordination. Integral transforms Spec. Funct., Vol. 18, pp. 207-216.

[11]Ruscheweyh St., (1981). Neighbourhoods of univalent functions, Proc.Amer. *Math.* Soc. 81, 521-527.

[12]N.D.Sangle and P.S. Mane (2019) "On New Subclasses of Analytic and Univalent Functions", IJRAR, Vol 6, Issue, 032 1 pp 134-141.

[13]S.B.Joshi and N.D. Sangle "New Subclasses of Univalent Functions Defined by using Generalized Salagean Operator",J.Indones Math.Sec.(MIHMI) Vol 15, O. 2(2009), pp 79-89.

.