IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

Vol. 7, Issue 10, October 2020

DOI 10.17148/IARJSET.2020.71020

Diameter of Circulant Graph $C_{n,r}$

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Abstract: Communication is a critical issue in the design of a parallel and distributed system. The speed of communication of an interconnection network is related to its diameter. The diameter is a measure of efficiency for studying the effects of link failures of a network with maximum time-delay or signal degradation. In this article, we determine the diameter of circulant graphs $C_{n,r}$.

Keywords: Diameter; Circulant Graph; Network Reliability; Signal Degradation.

I. Introduction

An interconnection network connects the processors of a parallel and distributed system. The topological structure of a network can be modelled by a connected graph whose vertices and edges represent the sites and communication links of a network, respectively. Many graph theoretic techniques can be used to study the efficiency and reliability of anetwork, as discussed in [1]-[6]. The diameter is a measure of efficiency for studying the effects of link failures of networks with maximum time-delay or signal degradation. The circulant graphs has many applications in wireless networks. In this article, we determine the diameter of circulant graphs $C_{n,r}$.

Definition 1.1 Let $C_n$be a cycle of $n$vertices. A *circulant graph*, denoted by $C_{n,r}$, is the graph with vertex set is same as of $V(C_n)$ and two vertices $u, v$are adjacent $C_{n,r}$if $u{\sim}v$in $C_n$or they are of distance $r$in C_n.

Proposition 1.1 In an n length cycle C_n following are true

(a) $d_{C_n}(x_i, x_j) = min\{|i-j|, n-|i-j|\}$

(b) $diam(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

The following lemma can be proved easily from the definition of $C_{n,r}$.

Lemma 1.1 The *circulant graph* $C_{n,r}$is regular and$d_{C_n}(x_i, x_j) = \left\lfloor \frac{d_{C_n}(x_i,x_j)}{r} \right\rfloor + \min\{m, r+1-m\}$ *where*

$$m = d_{C_n}(x_i, x_j) - r\left\lfloor \frac{d_{C_n}(x_i, x_j)}{r} \right\rfloor.$$

II. DIAMETER OF $C_{n,r}$

In literature, there exists no theoretical results of diameter of $C_{n,r}$. In the theorem below we give a formula for diameter of $C_{n,r}$explicitly.

Theorem 1 *Let* $diam(C_{n,r})$ be the diameter of circulant graph$C_{n,r}$. Then we have the following.

(a) For odd integer n

$$diam(C_{n,r}) = \begin{cases} \left\lfloor \frac{n}{2r} \right\rfloor - 1 + \left\lfloor \frac{n+1}{2} \right\rfloor, & if\ m = 0; \\ \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{r-2m}{2} \right\rfloor, & if\ \ 0 < m \le \left\lfloor \frac{r-1}{4} \right\rfloor; \\ \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{2r-2m+1}{2} \right\rfloor, & if\ \ \left\lfloor \frac{r}{2} \right\rfloor < m \le \left\lfloor \frac{3r-1}{4} \right\rfloor; \\ \left\lfloor \frac{n}{2r} \right\rfloor + 1 + \left\lfloor \frac{2m-r}{2} \right\rfloor, & if\ \ \left\lfloor \frac{3r-1}{4} \right\rfloor < m \le r-1. \end{cases}$$

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(b) For even integer n

$$diam(C_{n,r}) = \begin{cases} \left\lfloor \frac{n}{2r} \right\rfloor - 1 + \left\lfloor \frac{n+1}{2} \right\rfloor, & if\ m = 0; \\ \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{r-2m+1}{2} \right\rfloor, & if\ \ 0 < m \le \left\lfloor \frac{r-1}{4} \right\rfloor; \\ \left\lfloor \frac{n}{2r} \right\rfloor + r - m + 1, & if\ \ \left\lfloor \frac{r}{2} \right\rfloor < m \le \left\lfloor \frac{3r+1}{2} \right\rfloor; \\ \left\lfloor \frac{n}{2r} \right\rfloor + 1 + \left\lfloor \frac{2m-r+1}{2} \right\rfloor, & if\ \ \left\lfloor \frac{3r+1}{4} \right\rfloor < m \le r-1. \end{cases}$$

where, $m = \left\lfloor \frac{n}{2} \right\rfloor - r\left\lfloor \frac{n}{2r} \right\rfloor$.

Proof: We have found maximum eccentricity of the circulant graph. From the symmetricity the maximum eccentricity is attained by every vertex of $C_{n,r}$. So, without loss of generality, we find the eccentricity of x_0 i.e., we find a farthest distanced vertex of x_0. Let $u$be any vertex in $C_{n,r}$. Then $d_{C_{n,r}}(x_0,u) \le d_{C_n}(x_0,u)$ and it is true for every vertex $u$in $C_{n,r}$. Therefore, $\max_{u\in V(C_{n,r})} d_{C_{n,r}}(x_0,u) \le \max_{u\in V(C_n)} d_{C_n}(x_0,u)$ as $V(C_{n,r}) = V(C_n)$. Thus, we have $diam(C_{n,r}) \le diam(C_n)$ and hence $diam(C_{n,r}) \le \left\lfloor \frac{r}{2} \right\rfloor$. Also $2 \le r \le \left\lfloor \frac{r}{2} \right\rfloor$. So, by division algorithm, we get $m = \left\lfloor \frac{r}{2} \right\rfloor - r\left\lfloor \frac{n}{2r} \right\rfloor$ and $0 \le m \le r-1$. Let $S_i$be a sub-graph formed by $r+1$ vertices $x_{(i-1)r}, x_{(i-1)r+1}, \dots, x_{ir}$. As in $C_{n,r}$,every $r$distanced vertex are adjacent, so $S_i$is actually a cycle of r+1 vertices. As $\left\lfloor \frac{r}{2} \right\rfloor = r\left\lfloor \frac{n}{2r} \right\rfloor + m,\ \ 0 \le m \le r-1$. So, the path $P_{\left\lfloor \frac{n}{2} \right\rfloor}$from x_0 to $x_{\left\lfloor \frac{n}{2r} \right\rfloor}$ in $C_{n,r}$ is $S_1 \cup S_2 \cup \dots \cup S_{\left\lfloor \frac{n}{2r} \right\rfloor} \cup P_m$, where $P_m$is a path from $x_{r\left\lfloor \frac{n}{2r} \right\rfloor}$ to $x_{\left\lfloor \frac{n}{2} \right\rfloor}$. Clearly, here the maximum distanced vertex from x_0 will be in the sub-graph $S_{\left\lfloor \frac{n}{2r} \right\rfloor} \cup P_m$. We take followingtwo cases according as $n$is odd or even.

Case 1: ***$n$is odd.***

Sub-case (1a) :$m = 0$.In this case maximum distance from x_0 to $S_{\left\lfloor \frac{n}{2r} \right\rfloor}$ is $\left\lfloor \frac{n}{2r} \right\rfloor - 1 + \left\lfloor \frac{r+1}{2} \right\rfloor$ and $P_m$does not exist. So $diam(C_{n,r}) = \left\lfloor \frac{n}{2r} \right\rfloor - 1 + \left\lfloor \frac{r+1}{2} \right\rfloor$.

Sub-case (1b) :$0 < m \le \left\lfloor \frac{r}{2} \right\rfloor$. Since $r \ge 2m$, the r -th distanced vertex from $e$is $b$and it is in between of a and c. Now $d_{C_n}(c,e) = 2m+1$ and $d_{C_n}(b,e) = r$. So $d_{C_n}(b,c) = d_{C_n}(b,e) - d_{C_n}(c,e) = r - 2m - 1$ and $d_{C_n}(a,b) = d_{C_n}(a,c) - d_{C_n}(b,c) = 2m+1$. Hence maximum distance from x_0 to vertices of path from $a$to $b$is given by

$$d_{C_n}(x_0,a) + \left\lfloor \frac{d_{C_n}(a,b)+2}{2} \right\rfloor = \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{2m+1}{2} \right\rfloor$$

.

Again, maximum distance from x_0 to vertices of $b$to $c$path is given by

$$d_{C_n}(x_0,c) + \left\lfloor \frac{d_{C_n}(b,c)+1}{2} \right\rfloor = \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{r-2m}{2} \right\rfloor$$

.

Hence

$$diam(C_{n,r}) = \left\lfloor \frac{n}{2r} \right\rfloor + max\left\{\left\lfloor \frac{2m+1}{2} \right\rfloor, \left\lfloor \frac{r-2m}{2} \right\rfloor\right\}.$$

Thus, we have

$$diam(C_{n,r}) = \begin{cases} \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{r-2m}{2} \right\rfloor, & if\ 0 < m \le \left\lfloor \frac{r-1}{4} \right\rfloor; \\ \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{2m+1}{2} \right\rfloor, & if\ \left\lfloor \frac{r-1}{4} \right\rfloor < m \le \left\lfloor \frac{r}{2} \right\rfloor. \end{cases}$$

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Sub-case (1c): $\left\lfloor\frac{r}{2}\right\rfloor < m \leq (r-1)$. In this case r-th distanced vertex $b$from $e$will be in between of$c$and d. Also, r-th distanced vertex $f$from $b$will be in between of$a$and c. Here

$$d_{C_{n,r}}(c,b) = d_{C_n}(c,e) - d_{C_n}(b,e) = 2m+1-r$$

$$d_{C_{n,r}}(f,c) = d_{C_n}(f,b) - d_{C_n}(c,b) = 2r-2m-1$$

$$d_{C_{n,r}}(a,f) = d_{C_n}(a,c) - d_{C_n}(f,c) = 2m+1-r$$

$$d_{C_{n,r}}(b,d) = m_1 - d_{C_n}(c,b) = r-m-1.$$

So maximum distanced vertex from x_0 will lies in path $f-c$or in path $c-b$. Now maximum distance from x_0 to vertices of path $f-c$is $d_{C_{n,r}}(x_0,c) + \left\lfloor\frac{d_{C_n}(f,c)+2}{2}\right\rfloor = \left\lfloor\frac{n}{2r}\right\rfloor + \left\lfloor\frac{2r-2m+1}{2}\right\rfloor$ and maximum distance from x_0 to vertices of path $c-b$is $d_{C_{n,r}}(x_0,c) + \left\lfloor\frac{d_{C_n}(c,b)+1}{2}\right\rfloor = \left\lfloor\frac{n}{2r}\right\rfloor + \left\lfloor\frac{2m+2-r}{2}\right\rfloor$.
Hence

$$diam(C_{n,r}) = \left\lfloor\frac{n}{2r}\right\rfloor + max\left\{\left\lfloor\frac{2r-2m+1}{2}\right\rfloor, \left\lfloor\frac{2m+2-r}{2}\right\rfloor\right\}.$$

Thus

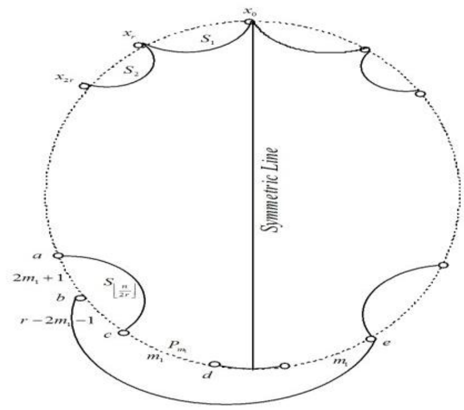

Fig 1: The graph $C_{n,r}$ with odd n and $\left\lfloor\frac{n}{2r}\right\rfloor < m \leq r-1$

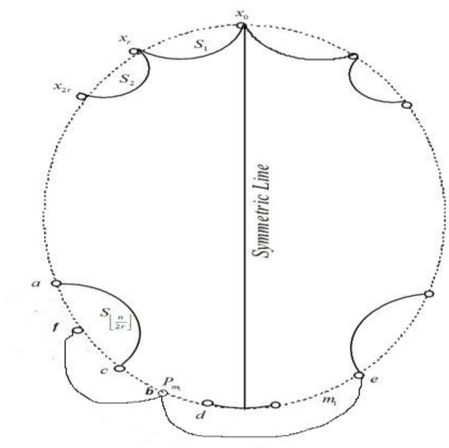

Fig 2: The graph $C_{n,r}$ with odd n and $0 < m \leq \left\lfloor\frac{n}{2r}\right\rfloor$

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$$diam(C_{n,r}) = \begin{cases} \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{2r-2m+1}{2} \right\rfloor, & if \left\lfloor \frac{r}{2} \right\rfloor < m \leq \left\lfloor \frac{3r-1}{4} \right\rfloor; \\ \left\lfloor \frac{n}{2r} \right\rfloor + 1 + \left\lfloor \frac{2m-r}{2} \right\rfloor, & if \left\lfloor \frac{3r-1}{4} \right\rfloor < m \leq r-1. \end{cases}$$

Case-II: n**is even.**

Sub-case (2a) :m=0.In this case maximum distance from x_0 to $S_{\left\lfloor \frac{n}{2r} \right\rfloor}$ is $\left\lfloor \frac{n}{2r} \right\rfloor - 1 + \left\lfloor \frac{r+1}{2} \right\rfloor$and P_m does not exist. So, in this case $diam(C_{n,r}) = \left\lfloor \frac{n}{2r} \right\rfloor - 1 + \left\lfloor \frac{r+1}{2} \right\rfloor$.

Sub-case (2b): $0 < m \leq \left\lfloor \frac{r}{2} \right\rfloor$. As, $r \geq m$so the r-th distanced vertex from $e$is$b$and it is inbetween $a$and c. Now $d_{C_n}(c,e) = 2m$ and $d_{C_n}(b,e) = r$. So $d_{C_n}(b,c) = d_{C_n}(b,e) - d_{C_n}(c,e) = r - 2m$ and $d_{C_n}(a,b) = d_{C_n}(a,c) - d_{C_n}(b,c) = 2m$. Hence maximum distance from x_0 to vertices of path from $a$to $b$is

$$d_{C_{n,r}}(x_0, a) + \left\lfloor \frac{d_{C_n}(a,b)+2}{2} \right\rfloor = \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{r-2m+1}{2} \right\rfloor.$$

Again, maximum distance from x_0 to vertices of $b$to$c$path is

$$d_{C_{n,r}}(x_0, c) + \left\lfloor \frac{d_{C_n}(b,c)+1}{2} \right\rfloor = \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{r-2m+1}{2} \right\rfloor.$$

Hence

$$diam(C_{n,r}) = \left\lfloor \frac{n}{2r} \right\rfloor + max\left\{m, \left\lfloor \frac{r-2m+1}{2} \right\rfloor\right\}.$$

Thus

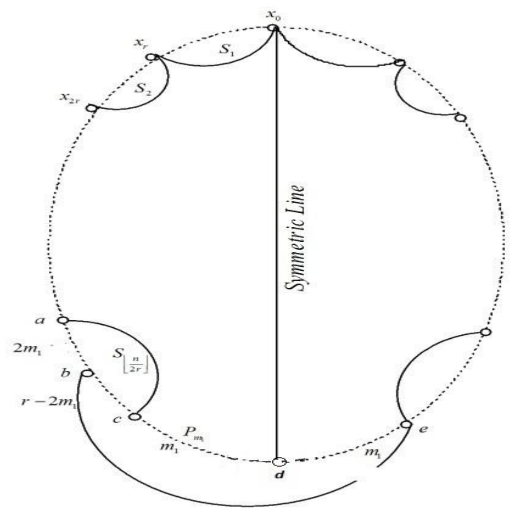

Fig 3: The graph $C_{n,r}$ with even n and $0 < m \leq \left\lfloor \frac{r}{2} \right\rfloor$

$$diam(C_{n,r}) = \begin{cases} \left\lfloor \frac{n}{2r} \right\rfloor + r - m + 1, & if \left\lfloor \frac{r}{2} \right\rfloor < m \leq \left\lfloor \frac{3r+1}{4} \right\rfloor; \\ \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{2m+1}{2} \right\rfloor, & if \left\lfloor \frac{3r+1}{4} \right\rfloor < m \leq r-1. \end{cases}$$

Sub-case (2c): $\left\lfloor \frac{r}{2} \right\rfloor < m \leq (r-1)$. In this case r-th distanced vertex $b$from $e$will be in between of$c$and d. Also r-thdistanced vertex $f$from $b$will be in between of$a$and c.
Here

$$d_{C_{n,r}}(c,b) = d_{C_n}(c,e) - d_{C_n}(b,e) = 2m - r$$

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$$d_{C_{n,r}}(f,c) = d_{C_n}(f,b) - d_{C_n}(c,b) = 2r - 2m$$

$$d_{C_{n,r}}(a,f) = d_{C_n}(a,c) - d_{C_n}(f,c) = 2m - r$$

$$d_{C_{n,r}}(b,d) = m - d_{C_n}(c,b) = r - m.$$

So maximum distanced vertex from x_0 will lie in path $f - c$or in path $c - b$. Now maximum distancefrom x_0 to vertices of path $f - c$is $d_{C_{n,r}}(x_0, c) + \left\lfloor \frac{d_{C_n}(f,c)+2}{2} \right\rfloor = \left\lfloor \frac{n}{2r} \right\rfloor + r - m + 1$ and maximum distance from x_0 to vertices of path $c - b$is

$$d_{C_{n,r}}(x_0, c) + \left\lfloor \frac{d_{C_n}(c,b)+1}{2} \right\rfloor = \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{r-2m+1}{2} \right\rfloor.$$

Hence,

$$diam(C_{n,r}) = \left\lfloor \frac{n}{2r} \right\rfloor + max\left\{r - m + 1, \left\lfloor \frac{2m-r+1}{2} \right\rfloor\right\}.$$

Thus

$$diam(C_{n,r}) = \begin{cases} \left\lfloor \frac{n}{2r} \right\rfloor + r - m + 1, \ if \ \left\lfloor \frac{r}{2} \right\rfloor < m \leq \left\lfloor \frac{3r+1}{4} \right\rfloor; \\ \left\lfloor \frac{n}{2r} \right\rfloor + \left\lfloor \frac{2m-r+1}{2} \right\rfloor, if \ \left\lfloor \frac{3r+1}{4} \right\rfloor < m \leq r - 1. \end{cases}$$

On account of all cases describes in above we get the results.

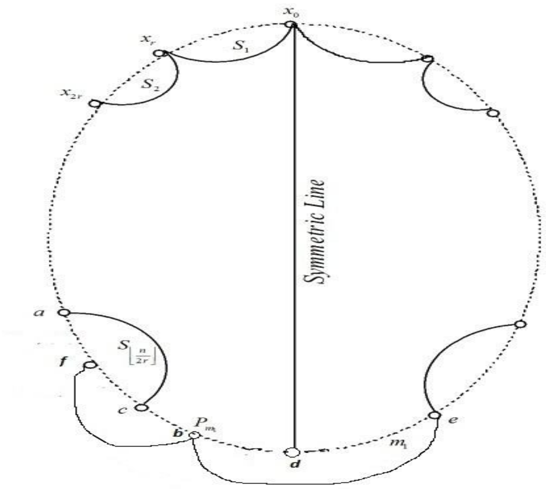

Fig 4: The graph $C_{n,r}$ with even n and $\frac{r}{2} < m \leq r - 1$

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