IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

Vol. 7, Issue 11, November 2020

DOI 10.17148/IARJSET.2020.71115

Metric Dimension of r-th power of paths

Laxman Saha

Assistant Professor, Department of Mathematics, Balurghat College, Balurghat 733101, India

Abstract: For a simple connected graph $G = (V, E)$, an ordered set $W \subseteq V$, is calleda *resolving set* of G if for every pair of two distinct vertices $u$and v, there is anelement $w$in Wsuch that $d(u,w) \neq d(v,w)$. A *metric basis* of G is a resolving setof G with minimum cardinality. The *metric dimension* of G is the cardinality ofa metric basis and it is denoted by $\beta(G)$. In this article, we determine the metricdimension of any power of finite paths.

Keywords: Code, Resolving set, Metric dimension.

I. INTRODUCTION

The study of Metric dimension or resolving set of a simple connected graph using distance between vertices is so much popular among research scholars as it is applicable to many areas like network, robotic navigation, drug design etc; The concept of the metric dimension of a graph was first introduced by Slater [13]. Theirintroduction of this invariantwas motivated by its application to the placement of a minimum number of sonar/lorandetecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. They use location set in place of resolving set. Also, Harary and Melter [8] in 1976 introduced the same concept as metric dimension rather than location number.

Throughout this article, $G = (V, E)$ denotes a simple connected graph with vertex set V and edge set E. Distance between two vertices u and v in G, denoted by $d(u, v)$ is thelength of a shortest $u - v$ path. For an ordered subset $W = \{w_1, w_2, \dots, w_k\} \subset V$ and a vertex v of G, *distance code of v with respect to S* is a k-vector given by

$$code_W(v) = \big(d(v, w_1), d(v, w_2), \dots, d(v, w_k)\big).$$

If $code_w(u) \neq code_w(v)$ for all distinct vertices u and v, then W is called a *resolving set* for the graph G. Every simple connected graph G has a resolving set as the vertex set V forms a resolving set. The *metric dimension* of graph G is the minimum cardinalityof a resolving set for G and it is denoted by $\beta(G)$. A resolving set with cardinality $\beta(G)$ is called a *metric basis* and elements of it are called *basis elements*. Throughout this article, distance code $code_w(v)$ is simply denoted by $code(v)$.

The r-th power of a graph G is the graph G^r having the vertex set same as that of G and edges between pair of vertices at distance at most r in G. Power graphs are usedto increase the connectivity of an existing network so it is necessary to study the metricdimensions of graphs and its power graphs. In[10] , Javaid et al. initiated to find themetric dimension for the square of cycles C_n^2. Imran et al. [9] later bounded the metricdimension of C_n^2 and C_n^2, and then Borchert and Gossel in [1] extended their results and determined the exact metric dimension of these two families of power of cycles.

Theorem 1.1 [1] For an integer$n > 6$

$$\beta(C_n^2) = \begin{cases} 4, & n \equiv 1(mod\ 4); \\ 3, & otherwise. \end{cases}$$

Theorem 1.2 [1] For an integer $n > 8$

$$\beta(C_n^3) = \begin{cases} 5, & n \equiv 1(mod\ 6); \\ 4, & otherwise. \end{cases}$$

In this article, we determine the metric dimension of power of any paths.

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588
International Advanced Research Journal in Science, Engineering and Technology
Vol. 7, Issue 11, November 2020
DOI 10.17148/IARJSET.2020.71115

II. PRELIMINARIES

In this section, we give some basic definitions, lemmas and proposition that will be used in sequel. For a path P_n with vertex set $V(P_n) = \{v_1, v_2, \ldots, v_{n-1}\}$, *r-th power of path* P_n, denoted by P_n^r, is a simple graph with vertex set $V(P_n)$ and two vertices v_i and v_j are adjacent if $d_{P_n}(v_i, v_i) \leq r$. It is clear to observe that $d_{P_n^r}(v_i, v_j) = \left\lceil \frac{d_{P_n}(v_i,v_j)}{r} \right\rceil = \left\lceil \frac{|v_i - v_j|}{r} \right\rceil = \left\lceil \frac{|i-j|}{r} \right\rceil$.The set $V(P_n^r)$ (vertex set of P_n^r) can be divided into the *blocks* $B_0, B_1, \ldots, B_{\lceil \frac{n}{r} \rceil - 1}$where the blocks B_i*'s* are defined by $B_i = \{v_{ir+j} \in V(P_n^r) : 0 \leq j \leq (r-1)\}$.

Proposition 2.1 *For* P_n^r, *the blocks* B_i*'s* have the following properties

a) Each block B_i with $0 \leq i \leq \left\lceil \frac{n}{r} \right\rceil - 1$, consists $r$elements, namely,$v_{ir}, v_{ir+1}, v_{ir+2}, \ldots, v_{ir+(r-1)}$ whereas the block $B_{\lceil \frac{n}{r} \rceil} - 1$ contain exactly ℓ elements provided $n \equiv \ell \ mod\ (r)$.

b) For each $i \in \left\{0, 1, \ldots, \left\lceil \frac{n}{r} \right\rceil - 1\right\}$, the induced sub-graph of B_i forms a *clique*.

Lemma 2.1 Any $r + 1$ consecutive vertices in P_n^r forms a clique.

Proof: Recall that $V(P_n^r) = \{v_0, v_1, \ldots, v_{n-1}\}$ and two vertices v_i and v_j are adjacent in P_n^rif and only if $|i - j| \leq r$. Let $S = \{v_i, v_{i+1}, \ldots, v_{i+r}\}$ be a set of $r + 1$ consecutive vertices in P_n^r. Also let v_p and v_q be any two vertices in S. Since $v_p, v_q \in S, p, q \in \{i, i+1, \ldots, r\}$ and hence $|p - q| \leq r$, which imply that v_p and v_q are adjacent. Therefore, any pair of vertices in S are adjacent $i.e.$, S forms a clique in P_n^r.

Definition 2.1 For an integer i satisfying $0 \leq i \leq r - 1$, by a $class[i]$ we mean the set $[i] = \{j \equiv i \ (mod\ r), 0 \leq j \leq n - 1\}$. For $0 \leq t \leq r - 1$, a vertex v_k is called t*-class* element if $k \in [t]$. Also, a vertex v_k is called the largest t*-class* element if k is the largest element in $[t]$. From here to onward, we denote the set of all *t-class* elements by $S[t]$.

Lemma 2.2 Let $r \geq 2$ be an integer and $\emptyset$ be the empty set. Then following are hold in P_n^r

(a) $S_{[x]} = S_{[y]}$ for $x \equiv y \ (mod\ r)$

(b) $S_{[x]} \cap S_{[y]} = \emptyset$ for $x \not\equiv y \ (mod\ r)$.

(c) $V(P_n^r) = \cup_{t=0}^{r-1} S[t]$.

Proof : (a) From definition, it is clear that $S[x] = S_{[y]}$for $x \equiv y \ (mod\ r)$.

(b) If possible, let $x \not\equiv y \ (mod\ r)$ and $v_k \in S[x] \cap S[y]$ for some k. Then $k \in [x]$ and $k \in [y]$ $i.e.$, $k \in [x] \cap [y]$, which is a contradiction as $x \not\equiv y \ (mod\ r)$ implies $[x] \cap [y] = \emptyset$. Therefore, $S_{[x]} \cap S_{[y]} = \emptyset$ whenever $x \not\equiv y \ (mod\ r)$.

(c) From definition of t-class element, we have $\cup_{t=0}^{r-1} S[t] \subseteq V(P_n^r)$. Now we show the reverse condition $V(P_r^n) \subseteq \cup_{t=0}^{r-1} S[t]$. Let $v_k \in V(P_n^r)$ be an arbitrary vertex. Then there exists an unique integer t such that $k = t \ (mod\ r)$ and $0 \leq t \leq r - 1$. Then $v_k \in S_{[t]}$because an element $v_k \in S_{[t]}$ if and only if $k \in [t]$. Therefore $v_k \in \cup_{t=0}^{r-1} S_{[t]}$. This completes the proof.

Lemma 2.3 For any two vertices $u = v_{ir+r_1}$and $w = v_{ir+r_2} of\ P_n^r$,

$$d_{P_n^r}(u, v) = \begin{cases} |i - j|, & if \quad r_2 \leq r_1; \\ |i - j| + 1, & otherwise. \end{cases}$$

Proof: Without loss of generality, we may assume u is left to w $i.e.$, $ir + r_1 < jr + r_2$. If $r_2 \leq r_1$, then

$$d(u, v) = \left\lceil \frac{jr + j_1 - ir - i_1}{r} \right\rceil$$

IARJSET

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

Vol. 7, Issue 11, November 2020

DOI 10.17148/IARJSET.2020.71115

$$= \left\lceil \frac{(j-i)r + j_1 - i_1}{r} \right\rceil$$
$$= \left\lceil \frac{(j-i-1)r + \{r-(i_1-j_1)}{r} \right\rceil = j - i$$

Again if $r_2 > r_1$, then $d(u,v) = \left\lceil \frac{(j-i)r+(j_i-i_1)}{r} \right\rceil = j - i + 1$.

Remark 2.1 For $0 \le s \le r-1, d(v_s,u) = d(v_s,w)$ implies $d(v_{ir+s},u) = d(v_{ir+s},w)$.

III. Metric Dimension of P_n^r

In this section, first we present a lower bound for the metric dimension of P_n^r and then we build up a resolving set with cardinality same as that lower bound. It is noted that $\{v_i, v_{i+1}, \ldots, v_{i+r}\}$ forms a clique in P_n^r for every $i \in \{0,1,\ldots,n-r-1\}$. The lemma below [4] represents an effective result to determine a lower bound for the metric dimension of P_n^r.

Lemma 3.1 Let $A \subset \{v_i, v_{i+1}, \ldots, v_{i+r}\}$ with $|A| = \ell,\ for\ 2 \le l \le r+1$. If X resolves A then $|X| \ge \ell - 1$.

Proof. We prove this lemma by induction on ℓ. The result obviously is true for $\ell = 2$. Now we show that it is true for $\ell = k$ with an assumption that it holds for $\ell = k-1$. Let the elements of A in orderare $v_{i+a_1}, v_{i+a_2}, \ldots, v_{i+a_k}$, where $0 \le a_1 < a_2 < \ldots < a_k \le r$. As X resolves A, there is an element $x \in X$ such that $d(x, v_{i+a_1}) \ne d(x, v_{i+a_2})$. If $x = v_{i+a_2}$, then $d(x, v_{i+a_s}) = 1$ for all $s \in \{1,3,\cdots,k\}$ and $X \setminus \{x\}$ resolves $A \setminus \{v_{i+a_2}\}$. If $x \ne v_{i+a_2}$, then $d(x, v_{i+a_2}) = d(x, v_{i+a_s})$ for all $s \in \{2,3,\cdots,k\}$ and $X \setminus \{x\}$ resolves $A \setminus \{v_{i+a_1}\}$. Thus in both cases $X \setminus \{x\}$ resolves a subset of $\{v_i, v_{i+1}, \ldots, v_{i+r}\}$ with cardinality $k-1$. Therefore, by assumption, $|X \setminus \{x\}| \ge k-2$ and hence $|X| \ge k-1$.

With the help of Lemma 3.1, the following result represents a lower bound for $\beta(P_r^n)$ for all values of n and $r < \frac{n}{2}$.

Lemma 3.2 For two integers $n\ and\ r$ with $r < \frac{n}{2},\ \beta(P_r^n) \ge r$.

Proof. Let B be a resolving set of P_r^n and $v_i \in B$. Then $A = \{v_i, v_{i+1}, \ldots, v_{i+r}\}$is not resolved by v_i as $d(v_i, v_{i+s}) = 1$ for all $s \in \{1,2,\cdots,r\}$. Applying Lemma 3.1, we have $|B \setminus \{v_i\}| \ge r-1$ and so$|B| \ge r$.

Now our aim is to construct a resolving set for P_r^n with cardinality r.

Lemma 3.3 Let i and ℓ be two integers such that $0 \le i < i + \ell \le r-1$. Then $A = \{v_i, v_{i+1}, \ldots, v_{i+\ell}\} \subset V(P_n^r)$ resolves the set $\bigcup_{j=1}^{\lceil \frac{n}{r} \rceil - 1} \{v_{jr+i}, \ldots, v_{jr+i+\ell+1}\} \subseteq V(P_n^r)$, *i.e.*, A resolves the set $\bigcup_{j=i}^{i+\ell+1} S[j]$

Proof. To prove this lemma, it is sufficient to show that for any pair of vertices $u, w \in \bigcup_{j=1}^{\lceil \frac{n}{r} \rceil - 1} \{v_{jr+i}, \ldots, v_{jr+i+\ell+1}\}$ there exists at least one vertex $x \in A$ such that $d(u,x) \ne d(w,x)$. Let $u, w \in \bigcup_{j=1}^{\lceil \frac{n}{r} \rceil - 1} \{v_{jr+i}, \ldots, v_{jr+i+\ell+1}\}$ be two distinct vertices. Then we may write $u = v_{ar+r_1}$ and $w = v_{br+r_2}$ for some $0 \le a, b \le \left\lceil \frac{n}{r} \right\rceil - 1$ and $i \le r_1, r_2 \le i + \ell + 1$. Without loss of generality, we may assume u is in left side of w, i.e., $ar + r_1 < br + r_2$. Now if we take $x = v_{r_1} \in A$, then $d(v_{r_1}, u) = a$ and $d(v_{r_1}, w) = b$ or $b+1$ according as $r_2 \le r_1$ or $r_2 > r_1$. Therefore if $r_2 > r_1$, then $d(u, v_{r_1}) \ne d(w, v_{r_1})$. Again if $r_2 \le r_1$, then also $d(u, v_{r_1}) \ne d(w, v_{r_1})$ as $ar + r_1 < br + r_2$. So $v_{r_1} \in A$ resolves $u\ and\ w$; and hence A resolves $\bigcup_{j=1}^{\lceil \frac{n}{r} \rceil - 1} \{v_{jr+i}, \ldots, v_{jr+i+\ell+1}\}$.

By similar argument as in Lemma 3.3, we have the following result.

ISSN (Online) 2393-8021
ISSN (Print) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

Vol. 7, Issue 11, November 2020

DOI 10.17148/IARJSET.2020.71115

Lemma 3.4 Let i *and* ℓ be two integers such that $0 \leq i < i + \ell \leq r - 1$. Then $B = \{v_{n-r+i}, v_{n-r+i+1}, \ldots, v_{n-r+i+\ell}\} \subset V(P_n^r)$ resolves the set $\bigcup_{j=1}^{\lceil \frac{n}{r} \rceil - 1} \{v_{n-jr+i-1}, \ldots, v_{n-jr+i+\ell}\} \subseteq V\ (P_n^r)$, *i.e.*, B resolves the set $\bigcup_{j=i-1}^{i+l} S[j]$.

The following theorem gives the exact value of metric dimension of P_n^r for all values of n and r.

Theorem 1. For two integers n and r with $r < \frac{n}{2}$, $\beta(P_n^r) = r$.

Proof. From Lemma 3.2, we have $\beta(P_n^r) \geq r$. Thus, to prove the theorem it is sufficient to construct a resolving set A with cardinality r. Consider $A = \{v_0, v_1, \ldots, v_{r-1}\}$ be the set of first r consecutive vertices of P_n^r . Now if we apply Lemma 3.3 to A then A resolve $\bigcup_{t=0}^{r-1} S_{[t]} = V(P_n^r)$. Therefore A is resolving set for P_n^r with cardinality r and consequently $\beta(P_n^r) \leq r$. Hence the theorem.

REFERENCES

[1]. A. Borchert and S. Gosselin, The metric dimension of circulant graphs and Cayley hypergraphs. *Util. Math.* 2014.
[2]. G. Chartrand and P. Zhang, "The theory and applications of resolvability in graphs, A survey," *Congr. Numer.,* 160**,** 2003, p. 47–68.
[3]. M. A. Chaudhry, I. Javaid and M. Salman, "Fault-tolerant metric and partition dimensionof graphs," *Util. Math.,*83, 2010, p. 187-199.
[4]. C. Grigorious, T. Kalinowski, J. Ryan and S. Stephen, "The metric dimension of the circulant graph$C_n(1,2,3,4)$," *Australasian Journal of Combinatorics,* 69(3), 2017, p. 417–441.
[5]. C. Grigorious, P. Manuel, M. Miller, B. Rajan and S. Stephen, "On the metric dimension of circulant and harary graphs," *Appl. Math. Comput.,* 248, 2014, p. 4754.
[6]. F. Harary and R. A. Melter, "On the metric dimension of a graph," *Ars. Combin.,*2, 1976, p. 191-195.
[7]. C. Hernando, M. Mora, P. J. Slater and D. R. Wood, "Fault-Tolerant Metric Dimensionof Graphs," *Ramanujan Math. Society Lecture Notes.,*5, 2008, p. 81–85.
[8]. F. Harary and R. Melter, "On the metric dimension of a graph," *Ars Combinatoria*, 2, 1976, p. 191–195.
[9]. M. Imran, A. Baig, S. A. U. H. Bokhary and I. Javaid, "On the metric dimension ofcirculant graphs," *Appl. Math. Lett.,*25(3)**,** 2012, p. 320–325.
[10]. I. Javaid, M.T. Rahim and K. Ali, "Families of regular graphs with constant metric dimension," *Util. Math.,*75,2008,p. 21–33.
[11]. I. Javaid, M. Salman, M. A. Chaudhry and S. Shokat, " Fault-Tolerance in Resolvability," *Util. Math.,* 80, 2009, p. 263–275.
[12]. S. Khuller, B. Raghavachari and A. Rosenfeld, "Landmarks in graphs,"*Discrete Appl.Math.,*70 (3), 1996, p. 217–229.
[13]. P. J. Slater, "Leaves of trees," *Congr. Numer.*, 14**,** 1975, p. 549-559.