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Mathematical Problem for the N bodies

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Abstract: A method for analytical solution of the problem for N material points in a stationary central force field is proposed. The method is based on the solution of the problem for two bodies and the feature of the center of mass (CM) of a system of material points.

Keywords: problem for N bodies.

I. INTRODUCTION

1. Problem for one body.

In physics, the problem of a body (material point) moving in a central stationary force field of the form: f (r), where r is the distance between the center of the field and the material point, has been known for a long time. In the case of a gravitational or electrostatic field (Fig. 1), the force acting on the material point has the form:

$$\vec{\mathbf{F}}(r) = -f(r)\vec{\mathbf{e}}_{\mathbf{r}} = -\frac{k}{r^2}\vec{\mathbf{e}}_{\mathbf{r}} \quad (1)$$

where k is a coefficient of proportionality depending on the source of the field and the body, and $\vec{\mathbf{e}}_{\mathbf{r}}$ is the unit vector with direction from the center of the field, in which the beginning of the coordinate system is selected, to the material point. When f(r) > 0 we have the force of attraction, and when f(r) < 0 - the force of repulsion.

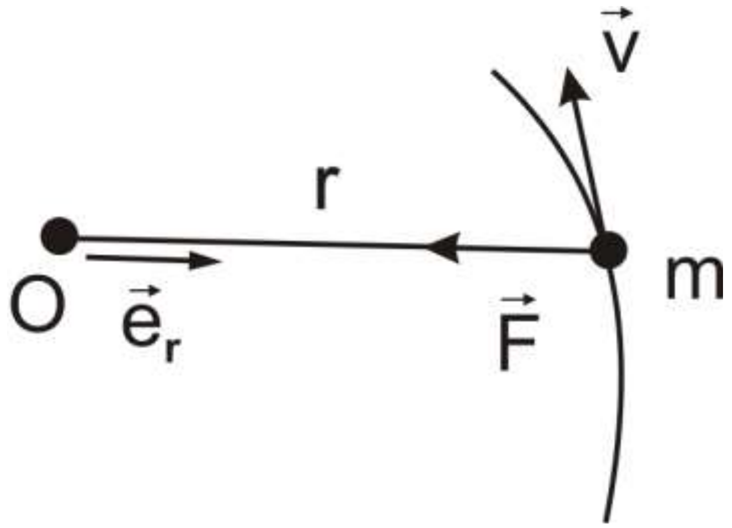

Fig.1: Motion of a material point in a stationary central force field

The differential equation describing the motion of a material point in field [1] is:

$$m\frac{d^2\vec{\mathbf{r}}}{dt^2} = -f(r)\vec{\mathbf{e}}_{\mathbf{r}} \quad (2)$$

The solution of this simple second-order differential equation, i. e. the law of motion $\vec{\mathbf{r}} = \vec{\mathbf{r}}(t)$ has long been known [1]. Depending on the initial conditions (at t = t_0, $\vec{\mathbf{r}}(t_0) = \vec{\mathbf{r}}_{\mathbf{0}}$, $\vec{\mathbf{v}}(t_0) = \vec{\mathbf{v}}_{\mathbf{0}}$) and the total mechanical energy E of the point, the trajectory is a conical section: ellipse (circle), parabola, hyperbola or in a special case a straight line, as the motion can be finite at E < 0 or infinite at E ≥ 0.

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2. Problem for two bodies.

The problem of a closed system of two bodies (material points) that interact with each other with central forces of the above type has been known for a long time too:

$$\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21} = f(r)\vec{\mathbf{e}}_{\mathbf{r}} \quad (3)$$

Their movement around the common center of mass can be found as follows [1].

Consider a system of two material points. We shall work in the system of CM. In order to describe their motion, we shall introduce a vector (Fig. 2):

$$\vec{\mathbf{r}}_{12} = \vec{\mathbf{r}}_{\mathbf{2c}} - \vec{\mathbf{r}}_{\mathbf{1c}} \quad (4)$$

which determines the position of the second point relative to the first. We shall translate this vector so that its beginning is in the center of mass C.

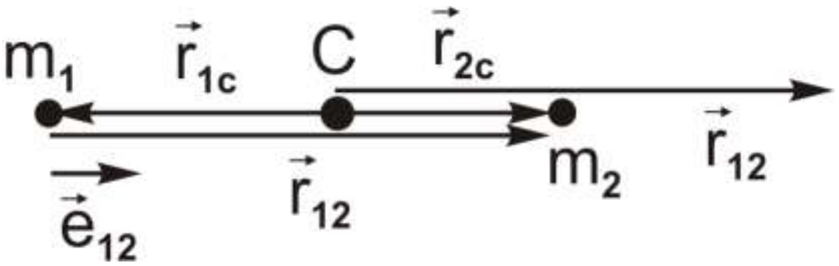

Fig.2: Problem for two bodies.

For the center of mass C we get:

$$\vec{\mathbf{r}}_{\mathbf{c}} = \frac{m_1\vec{\mathbf{r}}_{\mathbf{1c}} + m_2\vec{\mathbf{r}}_{\mathbf{2c}}}{m_1 + m_2} = 0 \quad (5)$$

From these two equations we can find:

$$\vec{\mathbf{r}}_{\mathbf{1c}} = -\frac{m_2}{m_1 + m_2}\vec{\mathbf{r}}_{12} \qquad \vec{\mathbf{r}}_{2c} = \frac{m_1}{m_1 + m_2}\vec{\mathbf{r}}_{12} \quad (6)$$

For the equations of motion for the two points we have:

$$\frac{d^2\vec{\mathbf{r}}_{\mathbf{1c}}}{dt^2} = \frac{f(r_{12})}{m_1}\vec{\mathbf{e}}_{\mathbf{12}} \quad (7)$$

$$\frac{d^2\vec{\mathbf{r}}_{\mathbf{2c}}}{dt^2} = -\frac{f(r_{12})}{m_2}\vec{\mathbf{e}}_{\mathbf{12}} \quad (8)$$

If we subtract the first equation (7) from the second (8) we obtain:

$$\frac{d^2\vec{\mathbf{r}}_{\mathbf{12}}}{dt^2} = \frac{d^2\vec{\mathbf{r}}_{\mathbf{2c}}}{dt^2} - \frac{d^2\vec{\mathbf{r}}_{\mathbf{1c}}}{dt^2} = -\frac{f(r_{12})}{m_2}\vec{\mathbf{e}}_{\mathbf{12}} - \frac{f(r_{12})}{m_1}\vec{\mathbf{e}}_{\mathbf{12}} = -\left(\frac{1}{m_2} + \frac{1}{m_1}\right)f(r_{12})\vec{\mathbf{e}}_{\mathbf{12}} = -\frac{1}{\mu}f(r_{12})\vec{\mathbf{e}}_{\mathbf{12}} \quad (9)$$

The equation (9) can be formally considered as an equation of motion of an imaginary point with mass $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$, called reduced mass, determined by the condition $\dfrac{1}{\mu} = \dfrac{1}{m_1} + \dfrac{1}{m_2}$.

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The problem for both bodies is reduced to the motion of an imaginary particle in a central force field centered in C, with a law of motion $\vec{\mathbf{r}}_{12}(t)$ determined by the above equation. Finding this law by solving the differential equation of the second order under certain initial conditions (at t = t_0, $\vec{\mathbf{r}}_1(t_0)=\vec{\mathbf{r}}_{01}$, $\vec{\mathbf{v}}_1(t_0)=\vec{\mathbf{v}}_{01}$, $\vec{\mathbf{r}}_2(t_0)=\vec{\mathbf{r}}_{02}$, $\vec{\mathbf{v}}_2(t_0)=\vec{\mathbf{v}}_{02}$), from the above equations we can find the laws of motion of the two particles: $\vec{\mathbf{r}}_{1c}(t)$ и $\vec{\mathbf{r}}_{2c}(t)$. The trajectories of the two points are conjugated between them conic sections, depending on the total energies E_1 and E_2, as the motion can be finite or infinite. The center of the masses always remains between the two points on the straight line that connects them. Note that we could achieve the same result if we assume, for example, that: $m_1 \to \infty$. Then: $\mu \to m_2$, $\vec{\mathbf{r}}_{1c} \to 0$, $\vec{\mathbf{r}}_{2c} \to \vec{\mathbf{r}}_{12}$ and $m_2 \dfrac{d^2\vec{\mathbf{r}}_{2c}}{dt^2} = -f(r)\vec{\mathbf{e}}_r$ and the second point would move around the center of mass C, determined by the first point – i. e. we come to the problem of one body.

II. PROBLEM FOR N BODIES.

Consider a closed system of N material points (N ≥ 3), between which internal forces of the type (1) act:

$$\vec{\mathbf{F}}_{ij} = -\vec{\mathbf{F}}_{ji} = f(r)\vec{\mathbf{e}}_{ij} \qquad (\mathrm{i,j}=1,2,...,\mathrm{N};\mathrm{i}\neq \mathrm{j})$$

In this case, the problem of finding the laws of motion and velocity of individual points (except in some special cases) is not solved in a general analytical form. In practice, numerical calculations of the differential equations of motion are applied in order to obtain the necessary results.

We shall propose a method for solving the problem for N bodies, based on the problem for 2 bodies and the properties of the center of masses.

1. Problem for 3 bodies.

In order to better understand the method of solving the problem for N bodies, we will first consider the problem for 3 bodies.

1.1. Basic theorem.

Let us consider a closed system of 3 material points with masses m_1, m_2, m_3 and radius vectors $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2, \vec{\mathbf{r}}_3$ (Fig. 3). The three points lie in one plane. The center of mass C of the system is known:

$$\vec{\mathbf{r}}_c = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2 + m_3\vec{\mathbf{r}}_3}{m_1 + m_2 + m_3} \quad (10)$$

and the centers of mass of each pair of points in the system separately C_{12}, C_{13}, C_{23} are also known:

$$\vec{\mathbf{r}}_{12} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2} \qquad \vec{\mathbf{r}}_{13} = \frac{m_1\vec{\mathbf{r}}_1 + m_3\vec{\mathbf{r}}_3}{m_1 + m_3} \qquad \vec{\mathbf{r}}_{23} = \frac{m_2\vec{\mathbf{r}}_2 + m_3\vec{\mathbf{r}}_3}{m_2 + m_3} \quad (11)$$

We shall prove the following theorem:

Theorem: Each of the centers of mass of each pair of points, the center of mass of the whole system C and the unpaired material point lie in a straight line.

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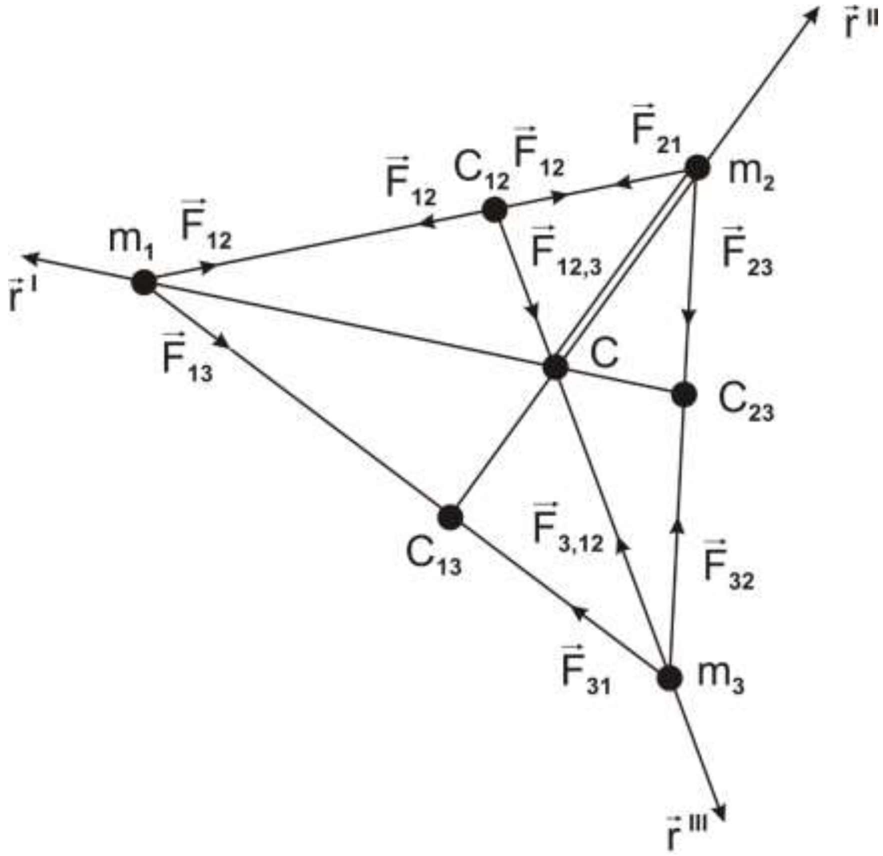

Fig.3: Basic theorem and problem for three bodies

Proof: Consider the three points C_{12}, C and m_3. According to the properties of the center of mass we shall have:

$$(m_1 + m_2 + m_3)\vec{\mathbf{r}}_{\mathbf{c}} = (m_1 + m_2)\vec{\mathbf{r}}_{\mathbf{12}} + m_3\vec{\mathbf{r}}_{\mathbf{3}}$$

If we choose the reference system with beginning at point C_{12} we have: $\vec{\mathbf{r}}_{\mathbf{12}} = 0$ and from the above equation (10) we obtain:

$$(m_1 + m_2 + m_3)\vec{\mathbf{r}}_{\mathbf{c}} = m_3\vec{\mathbf{r}}_{\mathbf{3}} \quad (11)$$

where $\vec{\mathbf{r}}_{\mathbf{c}}$ is the radius vector of C beginning at C_{12}. From the last equation (see eq.11) follows that vectors $\vec{\mathbf{r}}_{\mathbf{c}}$ and $\vec{\mathbf{r}}_{\mathbf{3}}$ are collinear with each other:

$$\vec{\mathbf{r}}_{\mathbf{c}} = \frac{m_3}{m_1 + m_2 + m_3}\vec{\mathbf{r}}_{\mathbf{3}} \quad (12)$$

and since they have a common point C_{12}, it follows that the points C_{12}, C and m_3 lie on a straight line.

Similarly, choosing the coordinate system to have beginning in C_{13} and in C_{23} consequently, we show that the other two triplets of points C_{13}, C, m_2 and C_{23}, C, m_1 also lie in straight lines.

Consequences of the theorem:

a) If $m_1 = m_2 = m_3$, then the center of mass C of the system coincides with the intersection points of the medians of the triangle formed by the three material points.

b) If any of the masses (for example $m_2 \to \infty$), then the points C_{12}, C_{23} and C trend to the location of this point.

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1.2. Problem for 3 bodies.

The main idea is the following. We can formally replace each pair of points in the system with their center of mass C_{12}, C_{13}, C_{23}, which has the same properties, and consider this center and the remaining material point as a two-body system. In this way, the problem of 3 bodies is reduced to the problem of 2 bodies, which solution is clear.

We shall work in the system of CM. Then, if we consider C_{12} and m_3, we get:

$$\vec{\mathbf{r}}_{\mathbf{c}} = \frac{(m_1+m_2)\vec{\mathbf{r}}_{\mathbf{12}} + m_3\vec{\mathbf{r}}_3}{m_1+m_2+m_3} = 0 \quad (13)$$

This way we obtain:

$$(m_1+m_2)\vec{\mathbf{r}}_{\mathbf{12}} = -m_3\vec{\mathbf{r}}_{\mathbf{3}} \quad (14)$$

We introduce a vector:

$$\vec{\mathbf{r}}''' = \vec{\mathbf{r}}_{\mathbf{3}} - \vec{\mathbf{r}}_{\mathbf{12}} \quad (15)$$

with the beginning in C. From the above two equations (13 and 14) we determine:

$$\vec{\mathbf{r}}_{\mathbf{3}} = \frac{m_1+m_2}{m_1+m_2+m_3}\vec{\mathbf{r}}''' \qquad \vec{\mathbf{r}}_{12} = -\frac{m_3}{m_1+m_2+m_3}\vec{\mathbf{r}}''' \quad (17)$$

The forces acting between the points are central and depend only on the distance between them. For them:

$$\vec{\mathbf{F}}_{\mathbf{12}} = -\vec{\mathbf{F}}_{\mathbf{21}} \qquad \vec{\mathbf{F}}_{13} = -\vec{\mathbf{F}}_{31} \qquad \vec{\mathbf{F}}_{23} = -\vec{\mathbf{F}}_{32} \quad (18)$$

By replacing the material points m_1 and m_2 with C_{12}, we seem to combine them in their center of mass, as the forces $\vec{\mathbf{F}}_{\mathbf{12}}$ and $\vec{\mathbf{F}}_{\mathbf{21}}$ are mutually neutralized, as the forces $\vec{\mathbf{F}}_{12,3} = -\vec{\mathbf{F}}_{3,12} = f(r)\vec{\mathbf{e}}_{\mathbf{r}}$, acting between C_{12} and m_3. Here f(r) is the magnitude of these forces, r is the distance between C_{12} and m_3, and $\vec{\mathbf{e}}_{\mathbf{r}}$ is the unit vector with direction from C_{12} to m_3. The reason for this alignment is given us by the following: the subsystem of the two points with masses m_1 and m_2 can be considered by us as a closed system of two points, located in the field of the third point, which is external to it. Then the center of mass C_{12} (with mass $m_1 + m_2$), which represents the subsystem, and the third point form a closed system of two bodies.

Then for m_3 and C_{12} we can write the differential equations of motion:

$$\frac{d^2\vec{\mathbf{r}}_{\mathbf{3}}}{dt^2} = -\frac{1}{m_3} f(r)\vec{\mathbf{e}}_{\mathbf{r}} \quad (19)$$

$$\frac{d^2\vec{\mathbf{r}}_{12}}{dt^2} = \frac{1}{m_1+m_2} f(r)\vec{\mathbf{e}}_{\mathbf{r}} \quad (20)$$

Subtracting equation 19 from equation 20 we get:

$$\frac{d^2\vec{\mathbf{r}}'''}{dt^2} = \frac{d^2\vec{\mathbf{r}}_{\mathbf{3}}}{dt^2} - \frac{d^2\vec{\mathbf{r}}_{12}}{dt^2} = -\left(\frac{1}{m_3} + \frac{1}{m_1+m_2}\right) f(r)\vec{\mathbf{e}}_{\mathbf{r}} = -\frac{1}{\mu} f(r)\vec{\mathbf{e}}_{\mathbf{r}} \quad (21)$$

where $\mu = \dfrac{(m_1+m_2)m_3}{m_1+m_2+m_3}$ is the reduced mass of a hypothetical particle with radius-vector $\vec{\mathbf{r}}'''$, moving in the field of the central force.

From equation 21 the law of motion $\vec{\mathbf{r}}'''(t)$ can be found. Then from the above equations (see 19 and 20) we determine the laws $\vec{\mathbf{r}}_{\mathbf{3}}(t)$ and $\vec{\mathbf{r}}_{\mathbf{12}}(t)$ as well. The finding of $\vec{\mathbf{r}}_{\mathbf{12}}(t)$ yet is not enough to determine $\vec{\mathbf{r}}_1(t)$ and $\vec{\mathbf{r}}_2(t)$, because we have only one equation for two unknowns:

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$$(m_1 + m_2)\vec{\mathbf{r}}_{\mathbf{12}} = m_1\vec{\mathbf{r}}_{\mathbf{1}} + m_2\vec{\mathbf{r}}_{\mathbf{2}} \quad (22)$$

To determine $\vec{\mathbf{r}}_1(t)$ and $\vec{\mathbf{r}}_2(t)$, we repeat the whole procedure twice for the pairs C_{13}, m_2 and C_{23}, m_1. Accordingly, we obtain:

$$\vec{\mathbf{r}}'' = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_{\mathbf{13}} \qquad \vec{\mathbf{r}}' = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_{\mathbf{23}}$$

$$\vec{\mathbf{r}}_{\mathbf{2}} = \frac{m_1 + m_3}{m_1 + m_2 + m_3}\vec{\mathbf{r}}'' \qquad \vec{\mathbf{r}}_1 = \frac{m_2 + m_3}{m_1 + m_2 + m_3}\vec{\mathbf{r}}'$$

$$\frac{d^2\vec{\mathbf{r}}''}{dt^2} = -\frac{1}{\mu''} f(r)\vec{\mathbf{e}}_{\mathbf{r}} \qquad \frac{d^2\vec{\mathbf{r}}'}{dt^2} = -\frac{1}{\mu'} f(r)\vec{\mathbf{e}}_{\mathbf{r}}$$

$$\mu'' = \frac{(m_1 + m_3)m_2}{m_1 + m_2 + m_3} \qquad \mu' = \frac{(m_2 + m_3)m_1}{m_1 + m_2 + m_3}$$

where we determine: $\vec{\mathbf{r}}''(t)$ $\quad$ $\vec{\mathbf{r}}'(t)$

and: $\vec{\mathbf{r}}_{\mathbf{2}}(t)$ $\quad$ $\vec{\mathbf{r}}_1(t)$

Thus the problem is solved.

When $f(r) \sim \dfrac{1}{r^2}$, the trajectories are conjugated conical sections determined by the total energy E_i of the particles, and the motion can be finite or infinite. Each point moves so that always the center of mass C remains on the straight line between it and the opposite center of mass C_{ij} of the other two points.

2. Problem for N bodies.

The proposed method in the previous paragraph can be summarized for the case of N bodies.

We Consider a closed system of N material points with masses m_i and radius vectors $\vec{\mathbf{r}}_{\mathbf{i}}$ (i = 1, ..., N) (Fig. 4). In the general case, the points of the system do not lie in one plane. The internal forces $\vec{\mathbf{F}}_{\mathbf{ij}} = -\vec{\mathbf{F}}_{\mathbf{ji}} = f(r_{ij})\vec{\mathbf{e}}_{\mathbf{ij}}$ act between them, which are functions only of the distance between the bodies. The common center of mass of the system we denote with C.

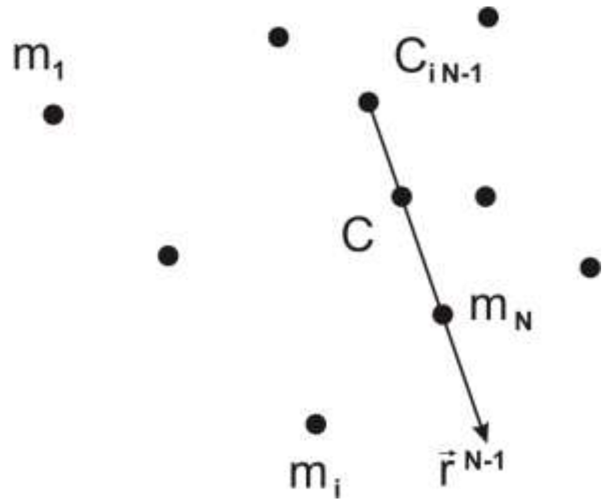

Fig.4: Problem for N bodies

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We consider an arbitrary point in the system (for example the point with number N). The remaining points form a subsystem with center of mass C_{iN-1} (i = 1, 2, ..., N - 1). The points C_{iN-1}, C and m_N lie on one straight line according to the proved theorem. Working in the system of CM, we can enter a vector:

$$\vec{\mathbf{r}}^{\mathbf{N}} = \vec{\mathbf{r}}_{\mathbf{N}} - \vec{\mathbf{r}}_{\mathbf{iN-1}} \quad (23)$$

with the beginning of C. Then the problem is reduced to the problem for 2 bodies, as we have combined all other points in their center of mass C_{iN-1} with a mass $m_{N-1} = \sum_{i=1}^{N-1} m_i$. This alignment is possible due to the third low of dynamics. Then the forces remain:

$$\vec{\mathbf{F}}_{\mathbf{iN-1,N}} = -\vec{\mathbf{F}}_{\mathbf{N,iN-1}} = f(r_{iN-1,N})\vec{\mathbf{e}}_{\mathbf{iN-1,N}}$$

We solve the following differential equation:

$$\frac{d^2\vec{\mathbf{r}}^{\mathbf{N}}}{dt^2} = -\frac{1}{\mu^{(N)}} f(r_{iN-1,N})\vec{\mathbf{e}}_{\mathbf{iN-1,N}} \quad (24)$$

and we find the law of motion $\vec{\mathbf{r}}^{\mathbf{N}}(t)$ of the imaginary point with reduced mass $\mu^{(N)}$. Then from the above equation we determine the law of motion of the point N: $\vec{\mathbf{r}}_{\mathbf{N}}(t)$.

Repeating this procedure N - 1 more times for every other point of the system (i = 1, ..., N - 1) and solving the same differential equation:

$$\frac{d^2\vec{\mathbf{r}}^{i}}{dt^2} = -\frac{1}{\mu^{(i)}} f(r_{i,j})\vec{\mathbf{e}}_{\mathbf{i},j} \qquad (\mathrm{i}=1,..., \mathrm{N}-1, \quad \mathrm{j} \neq \mathrm{i}) \quad (25)$$

but under different initial conditions, we also obtain the laws of motion of the other points $\vec{\mathbf{r}}_{\mathbf{i}}(t) \quad (\mathrm{i}=1,..., \mathrm{N}-1)$. With this the problem for N bodies is solved.

When: $f(r) \sim \frac{1}{r^2}$, the trajectories of the points are conjugated conical sections around the center of mass C, depending on the total mechanical energy E_i of each point, as the motion can be finite or infinite.

3. COMPARISON WITH PRACTICE

In astronomy, there are so-called globular star clusters in our Galaxy [2], which are self-gravitational systems of N stars (material points), moving in its common gravitational field. These systems are not closed because the stars are subjected to external gravitational forces by the Galaxy. Using the theorem for the decomposition of motion [1], we can assume that under the action of all external forces the center of mass C of a spherical star cluster moves as a material point, and the stars in it have motions around this center under the action of internal gravitational forces. In stellar astronomy, it has been shown [2] that the trajectories of stars inside a cluster around C can be conjugated conical sections with precision pericenters, as the type of section depending on the total mechanical energy E_i of the star. In most cases, $E_i < 0$ and the trajectories are either straight, passing through C, or ellipses (in a special case, circles). The movement is finite. As a result of stellar-star convergences (formally strikes), one of the stars may acquire a velocity greater than the parabolic one ($E_i > 0$), then its trajectory becomes a parabola or a hyperbola and the star dissipates from the cluster (infinite motion). It is also possible the opposite - a star, outside of the cluster, to enter in it and to change the movement of the other members. In the last two cases, the center of mass C of the cluster also changes.

These results from observations are in agreement with the results of the problem for N bodies.

II. CONCLUSION

In this article we proposed method for an analytical solution of the problem for N bodies in a stationary central force field. We can formally replace the real system with a two-body system form with an arbitrary point in the system and the center of mass of the rest (N-1) material points.

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