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A Quest for the Existence of Bi-Conditional Statements Over the Theory of a Model

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Abstract: Construction of the group of all bi-conditional statements over the theory of a model and discussing its properties and isomorphic forms for the quest and experiments of the existence of the converse of a statement over that model irrespective of its individual proof.

Keywords: Theory of a model, join as a logical connective, group of if statements, non-divisible infinite abelian group, infinite symmetric and permutation groups.

I. INTRODUCTION

In mathematics I think the most familiar and commonly asked question is that ' Is the converse holds?' . Yes its always being necessary to look after the existence of the converse statement after proving any theorem in mathematics. But it will always be a remarkable thing if for any statement we can tell that its converse holds over the respective theory or not without proving it. Here our quest and experiments are basically focused on that using some very basic group theoretic results.

If $\mathcal{U}$ is a model with ***card*** $\mathcal{U} \geq \aleph_0$ over the language $\mathcal{L}$, the theory of $\mathcal{U}$ is denoted by **Th**$\mathcal{U}$, is defined to be the set of all sentences of $\mathcal{L}$ (i.e formulas with no free variables) which are true in $\mathcal{U}$ with ***card*** $\mathcal{U} \geq \aleph_0$
So **Th** $\mathcal{U}$ = { σ of $\mathcal{L}$: $\mathcal{U} \models \sigma$} .Now let $\mathbf{\Omega} \subset$ **Th** $\mathcal{U}$ such that $\mathbf{\Omega}$ is the set of all $\sigma \in$ **Th** $\mathcal{U}$ whose converse are also in **Th** $\mathcal{U}$. We are denoting the converse of σ as σ^{-1}. So $\mathbf{\Omega} = \{ \sigma \mid \mathcal{U} \models \sigma \wedge \mathcal{U} \models \sigma^{-1}\}$.
Now its evident that $\sigma_i \in \Omega$ and $\sigma_j \in \Omega$ then $\sigma_i \wedge \sigma_j \in \Omega$. Actually our intension is to use '∧' as a binary composition on Ω .Without loss of generality ∧ is both commutative and associative. Now we treat an unique statement $\sigma \in \Omega$ as the identity statement and defined as σ = 'This is this' . So let $\sigma_i \in \Omega$ and $\sigma_i^{-1} \in \Omega$ be its converse then $\sigma_i \wedge \sigma_i^{-1} = \sigma \ \ \forall \sigma_i \in \Omega$. Here one thing of special attention is that we have not joined the statements by 'iff' rather than simply '&' . For example we can think $\sigma_i \in \Omega$, such that σ_i = this is X and σ_i^{-1} = X is this , then $\sigma_i \wedge \sigma_i^{-1} = \sigma$ = This is this. With the help of the underlying transitivity. In the same way we can treat other bi - conditional sentences also, for example $\mathbf{\Omega} \subset$ **Th** ℝ and let $\sigma_i \in \Omega$ where σ_i stands for 'closed and bounded subsets of ℝ are compact subsets of IR'. Now σ_i^{-1} i.e 'compact subsets of ℝ are closed and bounded subsets of ℝ' will also be in $\mathbf{\Omega}$. Now if we treat the clause 'closed and bounded subsets of IR' as 'This' and 'compact subsets of ℝ' as 'X' then we get σ_i = this is X, and σ_i^{-1} = X is this ;and evidently we get $\sigma_i \wedge \sigma_i^{-1} = \sigma$ = This is this . We will see all sentences like' Y is Y' or 'Y=Y' is as same as the identity statement σ = 'This is this'. So (Ω , ∧) be an abelian group. Also note that in (Ω , ∧) , if $\sigma_i \neq \sigma$ then $\sigma_i \wedge \sigma_i = \sigma_i^{\,2} \neq \sigma_i$ .From here we will do only some basic group theories and try to connect the desired results for further need of model theory.

Some special properties of the group (Ω , ∧):

1. (Ω , ∧) is torsion free according to its construction.
2. (Ω , ∧) is always infinite since its torsion free.
3. (Ω , ∧) is non trivial Abelian so (Ω , ∧) is not a free group.
4. (Ω , ∧) has no non trivial finite subgroup because its torsion free and abelian.
5. (Ω , ∧) is not finitely generated because by the theorem ' A finitely generated abelian group is free iff its torsion free'.
6. (Ω , ∧) is not divisible.

II. OBSERVATION

According to such properties there will not be any chance that (Ω , ∧) is similar to some of our known groups, even its not isomorphic to groups like (ℚ*,.) or (ℝ*,.) etc, according to o(***Th*** $\mathcal{U}$). Because they have finite subgroups. But there

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might be possibility that it can be similar with($\mathbb{Q}^*$/{-1,1},.) or(($\mathbb{R}^*$/{-1,1},.) etc. This observation leads us to go on our further quest and experiments.

Theorem: The *group Ω is isomorphic to a subgroup of the symmetric group of* ***Th*** $\mathcal{U}$.

Proof: Let Ω acts on **Th** $\mathcal{U}$. thus for each $\sigma_i \in \Omega$ we can get a bijection $\psi_{\sigma i} : \mathbf{Th}\ \mathcal{U} \rightarrow \mathbf{Th}\ \mathcal{U}$ defined as $\psi_{\sigma i}(a) = \sigma_i \wedge a$. Along with that there will be an associate homomorphism Σ: Ω → Sym(**Th** $\mathcal{U}$) , where Sym (**Th** $\mathcal{U}$) stands for the symmetric group of **Th** $\mathcal{U}$ with $\boldsymbol{\Sigma(\sigma_i)} = \psi_{\sigma i}$. So, = { $\sigma_i \in \Omega \mid \Sigma(\sigma_i) = \psi_I$ } where ψ_I be the identity map. Therefore by isomorphism theorem **Ω / Ker Σ** is isomorphic into Sym(**Th** $\mathcal{U}$) .Now claiming that Ker Σ = { σ } , because if $\sigma_i \in$ Ker Σ then $\Sigma(\sigma_i) = \psi_I$ where $\psi_I(a) = \sigma_i \wedge a = a$ $\forall a \in$ **Th** $\mathcal{U}$. So $\sigma_{i} = \sigma$. So Σ: Ω → Sym(**Th** $\mathcal{U}$) is an isomorphism.

Since **Ω** is abelian and **Σ** is not an onto isomorphism so **Σ(Ω)**is an ablian subgroup of Sym(**Th** $\mathcal{U}$) of **o(Ω)**. The above result is an immediate consequence of Cayley's theorem which states that every group is isomorphic to some permutation group.

Now our work will be to predict the exact structure of **Σ(Ω).** Because if we are able find any permutation $\boldsymbol{\psi_{\sigma i}}$ in **Σ(Ω)** then at a glance we are also able to say that $\boldsymbol{\sigma_i^{-1}}$ is also true in $\mathcal{U}$. We note this as a corollary,

***Corollary**: If $\psi_{\sigma i} \in \Sigma(\Omega)$ then $\mathcal{U} \vDash \sigma_i^{-1}$*

To focus on our central problem i.e on ***Th*** $\mathcal{U}$ if a statement is true then how to predict that its converse is also true or not without proving it; we will now electrify on the abelian subgroups of Sym(**Th** $\mathcal{U}$) *since* **Ω** is abelian and isomorphic to a subgroup of Sym(**Th** $\mathcal{U}$). Thus **Σ(Ω)** enjoys the same group theoretic properties as (**Ω** , ∧).

Its evident that Alt ***(Th*** $\mathcal{U}$) is obviously a normal subgroup of Sym(**Th** $\mathcal{U}$). But its simple also. Since **Σ(Ω)**is abelian and not finitely generated so Σ(Ω) can't be simple, thus there will be no question to compare Σ(Ω) and Alt ***(Th*** $\mathcal{U}$). Therefore we have to look about some else subgroups of Sym***(Th*** $\mathcal{U}$).

Let ***A*** be any infinite cardinal number , if φ ∈ Sym(**Th** $\mathcal{U}$) then the set S(φ) = { σ ∈ **Th** $\mathcal{U}$ | φ(σ) ≠ σ} Now we will define Sym(**Th** $\mathcal{U}$,*A)* = { φ ∈ Sym(**Th** $\mathcal{U}$) : o(S(φ)) < A } . It's easy to verify that Sym(**Th** $\mathcal{U}$,*A)* is a normal subgroup of Sym(**Th** $\mathcal{U}$). For our further experiments we will prefer A=$\aleph_0$.

***Theorem:** If G is an infinite subgroup of Sym(**Th** $\mathcal{U}$, $\aleph_0$) and k ∈ **Th** $\mathcal{U}$ then the order of the stabilizer of k is equals to the order of G.*

Proof: Let G_k = { x ∈ G | kx = x} and $o(G_k) \neq o(G)$ with all the above conditions. Now let T_k be the orbit of G containing k. So, o(G) = $o(G_k)o(T_k)$. Thus according to our assumption, $o(T_k) = o(G)$. (As cardinal arithmetic) . Now ∃ g∈G such that gk ≠ k . If b∈ T_k then ∃ h∈G such that kh =b . thus b∈ $S(h^{-1}gh)$ as $h^{-1}ghb \neq b$. Therefore, $T_k \subset \cup S(h^{-1}gh)$. Since each $S(h^{-1}gh)$ has same finite order say n and $o(T_k)=o(G)$, so g has o(G) numbers of conjugates. Also b∈ T_k then G_b is conjugate to G_k. Hence $o(G_b) < o(G)$. So there are n+1distinct elements $k_1,..,k_{n+1}$ of T_k . Since each conjugate of g and moves exactly n elements so they will be in some G_{ki} .Thus $o(G) \leq \Sigma(G_{ki}) = (n+1)o(G_k) < o(G)$, Which is a contradiction.

Corollary: *If G is an infinite subgroup of Sym(**Th** $\mathcal{U}$, $\aleph_0$) then there exist a proper sub group H of G such that o(H)=o(G).*

Corollary(ii): *If G is an infinite subgroup of Sym(**Th** $\mathcal{U}$, $\aleph_0$) and H is an abelian subgroup of G then there is an abelian subgroup K of G such that H⊂K and o(K)=o(G).*

Proof: By Zorn's lemma in G there will be a maximal abelian proper subgroup K such that *H⊂K. As we know that as G* is infinite and U⊂ *G and o(U)<o(G), then o(C(U))=o(G). Now if we put K=U* so we get *o(C(U))=o(G)* . Hence ∃ x∈ C(K)/K such that <K,x> be abelian ,which is a contradiction. So o(K)=o(G).

***Corollary (iia)**:If G is an uncountable Abelian group, then G has a well-ordered descending chain of subgroups of length o(G).*

We are skipping the proof here though we can find the proof in the following book*

#Thus if *Σ(Ω) or Ω are of order greater than* $\aleph_0$ then they have *a well-ordered descending chain of subgroups of length o(Ω).*

Lemma. *Let N be a nontrivial normal subgroup of Sym(**Th** $\mathcal{U}$).Then either N =Alt(**Th** $\mathcal{U}$) or N ≥ Sym(**Th** $\mathcal{U}$, $\aleph_0$).*
(i.e *N =Sym(**Th** $\mathcal{U}$, $\aleph_i$), for some i*∈IN)

We are skipping the proof here though we can find the proof in the following book**

Now let **o(*Th*** $\mathcal{U}$**)**$^+$ be the next higher cardinal after just o(***Th*** $\mathcal{U}$) , so we can see that *Sym(**Th**$\mathcal{U}$,* o(***Th***$\mathcal{U}$)$^+$) = *Sym(**Th*** $\mathcal{U}$).
Thus by the previous lemma we are getting-

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Lemma: *$\Sigma(\Omega)$ is not a normal subgroup of Sym(**Th $\mathcal{U}$**).*

Proof: its already verified that $\Sigma(\Omega) \neq$ Alt(**Th$\mathcal{U}$**) , again by our previous lemma its clear that Alt(**Th $\mathcal{U}$**) is also a normal subgroup of Sym(**Th $\mathcal{U}$**, $\aleph_0$). Which is also true for any c, such that $\aleph_0$<c< o(**Th $\mathcal{U}$**)$^+$. Since Alt(**Th $\mathcal{U}$**) is non abelian by corollary (ii).

#So we have to check other sub groups of *Sym(**Th $\mathcal{U}$**)* rather than Alt(**Th $\mathcal{U}$**) and Sym(**Th $\mathcal{U}$,b**)**,** with $\aleph_0 \leq$b< o(**Th $\mathcal{U}$**)$^+$.

III. CONCLUSION

Here we are halting our quest because the classification of infinitely generated abelian groups is far from complete, i.e at present it's an open problem. Though there is a complete classification of divisible abelian groups and torsion abelian groups but these are absolutely not fulfilled our present requirement. Though we had proceed to some filtered result and might be the help of computer algorithms and programs we can continue our further quest.

REFERENCES

1. Fundamentals of Model Theory by Willium Weiss and Cherie D'Mello (University of Toronto)
2. *Group theory by W.R. Scott (Dover Publication,New York)
3. ** Permutation Groups by John D. Dixon Brian Mortimer (Springer)
4. Maximal Abelian subgroups of the symmetric groups by J.Dixion (https://www.cambridge.org/core.)
5. (Notre Dame mathematical lectures 11) Ali Nesin - The Model theory of groups-University of Notre Dame Press (1989)