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Motion of Charged Particles in Uniform Electric Field and Magnetic Field.

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Abstract: If a charged particle accelerates then it produces both an electric field and a magnetic field. The interaction of particles can be described with the help of the concept of a field of force. When one particle acts on another then the particle creates a field around itself and a certain force then acts on every other particle located in this field. The study of the physical system consisting of charged particles in electromagnetic field constitute a major part of the whole of physics. Here considering the behaviour of a charged particles with some examples of uniform electric & magnetic fields in classical cases by starting with the general laws of classical electrodynamics in the covariant form can be considered.

Keywords: Electromagnetism, Lorentz Force, Covariant form, Gyro-radius, Cyclotron frequency, Helical path, cycloid.

I. OBJECTIVE OF STUDY

This study aimed for the following points:

- To know the common electromagnetic theories and its importance in industries.
- To observe some electromagnetic theories based on the objective effect of physics domain.
- To find the path of the charged particles in various fields.
- Not only electromagnetic theories used for functional theories, it is also widely used in manufacturing and research domain.

II. INTRODUCTION

Electromagnetism is a branch of physics deals with the study of the electromagnetic force, a type of physical interaction that occurs between electrically charged particles. Electromagnetic phenomena are defined in terms of that electromagnetic force, called as Lorentz force, which includes both electricity and magnetism as different manifestations of the same phenomenon. In an electric field a charged particle experiences a force and if this forces acting on any object are unbalanced then it will cause the object to accelerate. If two objects with same charge are brought towards each other then force produced will be repulsive and it will push them apart. In case of motion of a charged particle in a magnetic field, the magnetic force is perpendicular to the velocity of particle. The perpendicular force, qVXB acts as a centripetal force and produces a circular motion perpendicular to the magnetic field.

The electromagnetic force is carried by electromagnetic fields composed of electric fields and magnetic fields, and it is responsible for electromagnetic radiation such as light. The theoretical implications of electromagnetism describes the speed of light based on properties of the medium of propagation and led to the development of special relativity by Albert Einstein in 1905.[8]

Here the motion of a charged particles in various electromagnetic fields in the absence of a medium have been considered. First the general formulas are written down in the covariant form. Then the simple case of a uniform electric field, a uniform magnetic field, combined constant electric and magnetic fields are considered. The treatments of Landau-Lifshitz [2] and Jackson [1] have been taken . Consider a constant electromagnetic field, which does not depend on time. A constant electric field is determined by the components of $A^{\propto}$ (for non-relativistic case, considered as Scalar Potential) related to E by E = - grad ϕ , and by the vector potential A Where $\vec{H} = curl\,\vec{A}$

The equation of motion of a charge in an electromagnetic field can be written as

$$\frac{dp}{dt} = eE + \frac{e}{c} V * H \quad \text{……….1.1}$$

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The expression on the right of equation (1.1.) is called **Lorentz Force**. The first term gives the force which the electric field exerts on the charge does not depend on the velocity of the charge, and is along the direction of E. The second termt of which gives the force exerted by the magnetic field on the charge is proportional to the velocity of the charge and is directed perpendicular to the velocity and magnetic field $\vec{H}$.
The work done on the charged particle by the electric field is given by

$$\frac{d\varepsilon_{kin}}{dt} = eE.v \qquad \text{..........1.2}$$

The magnetic field does no work on a charge moving in it because the force which the magnetic field exerts is always perpendicular to the velocity of the charge.
Hence the energy of a charged particle in a constant time independent electromagnetic field can be written as

$$\varepsilon = \frac{mc^2}{\sqrt{1-\frac{\upsilon^2}{c^2}}} + e\phi \qquad \text{..........1.3}$$

The presence of the field adds to the energy of the particle i.e. eϕ the potential energy of the charge in the field. Energy depends only scalar but not on the vector potential. This means that the magnetic field does not affect the energy of the charge, only electric field can change the energy.[1].

III. GENERAL FORMULATION

The covariant form of the Maxwell's equations are –

$$\partial_\alpha F^{\alpha\beta} = -4\pi J^\beta \qquad \text{..........1.4}$$

and $$\partial_\alpha \tilde{F}^{\alpha\beta} = 0 \qquad \text{..........1.5}$$

where $$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \qquad \text{..........1.6}$$

$$\tilde{F}_{\alpha\beta} = \in_{\alpha\beta\gamma\sigma} F^{\gamma\sigma} \qquad \text{..........1.7}$$

and A_α, are the electromagnetic 4-vector potential and J^β is the four current density.
The covariant equation of motion of a charged particle can be written as :

$$m\frac{d^\alpha X^\alpha}{d\tau^2} = \frac{q}{c}\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)\frac{dX_\beta}{d\tau} \qquad \text{..........1.8}$$

Where τ is the proper time
These can be obtained by variational principle from action obtained Lagrangion density

$$L = \frac{-1}{4} G_{\alpha\beta} F^{\alpha\beta} - \frac{J_\alpha}{c} A^\alpha \qquad \text{..........1.9}$$

This gives rise to the conjugate momentum 4-vector.

$$P^\alpha = -m\frac{dx^\alpha}{d\tau} + \frac{q}{c} A^\alpha$$

..........1.10

4.1.MOTION OF CHARGED PARTICLES IN A CONSTANT UNIFORM (STATIC) ELECTRIC FIELD:

The equation of motion of a charged particle 'e' in a uniform constant electric field 'E' along x-axis on XY plane will be given by -

$$\ddot{P}_x = eE, \dot{P}_y = 0$$

So that, $$P_x = eEt, \dot{P}_y = P_0$$ ....1.11 Where P_0 be the momentum of the particle at the moment when $P_X = 0$.

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The kinetic energy of the particle becomes

$$\varepsilon_{kin} = c\sqrt{m^2c^2 + P} = \sqrt{m^2c^4 + c^2P_0^2 + (ceEt)^2}$$

or

$$\varepsilon_{kin} = \sqrt{\varepsilon_0^2 + (ceEt)^2} \qquad \text{……….1.12}$$

where ε_0 is the energy at t=0.

The relation between the energy, momentum and velocity of a charged particle is

$$V = \frac{Pc}{\varepsilon_{kin}}$$

or

$$\frac{dx}{dt} = \frac{c^2eEt}{\sqrt{\varepsilon_0^2 + (ceEt)^2}} \qquad \text{……….1.13}$$

on integrating we get

$$x = \frac{1}{eE}\sqrt{\varepsilon_0^2 + (ceEt)^2} \qquad \text{……….1.14}$$

similarly,

$$\frac{dy}{dt} = \frac{P_y c^2}{\varepsilon} = \frac{P_0 c^2}{\sqrt{\varepsilon_0^2 + (ceEt)^2}} \qquad \text{……….1.15}$$

which gives

$$y = \frac{P_0 c^2}{eE} = \sinh^{-1}\left(\frac{ceEt}{\varepsilon_0}\right) \qquad \text{……….1.16}$$

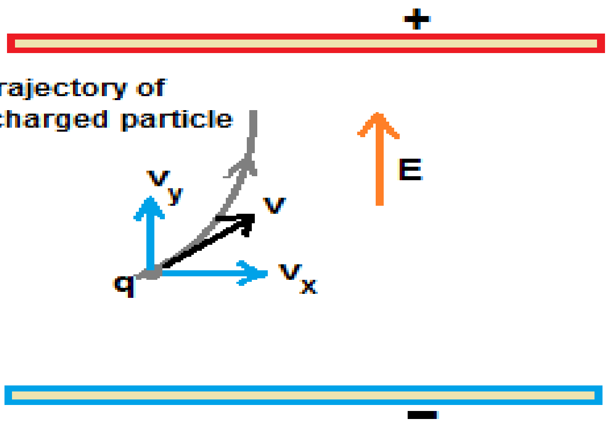
The equation of trajectory becomes

$$x = \frac{\varepsilon_0}{eE} = \cosh\frac{eEy}{P_0 c} \qquad \text{……….1.17}$$

Thus in a uniform electric field, a charge moves along a **catenary curve**.i.e it has U-like shape and similar to parabolic curve shown in figure-2.[1]

If the velocity of the charged particle is v << c , then that particle moves along a **parabola** given by expanding equation (1.17) in power series of $\frac{1}{c}$ and putting $P_0 = m\upsilon_0, \varepsilon_0 = mc^2$. Then

$$x = \frac{e\varepsilon}{2m\upsilon_0^2}y^2 + \text{constant} \qquad \text{……….1.18}$$



(Fig.-1shows trajctory of charged particle)

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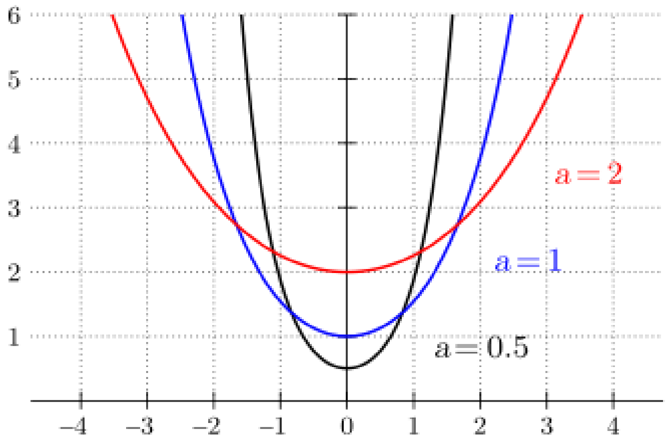

(Fig.-2 shows catenary curve)

4.2.MOTION OF CHARGED PARTICLES IN A CONSTANT UNIFORM MAGNETIC FIELD:
The equation of motion of a charged particle 'e' in a uniform magnetic field H along z-axis can be written as

$$\ddot{P} = \frac{e}{c}\vec{V} * \vec{H} \qquad \text{..........1.19}$$

Since energy of the particle is constant in magnetic field then by substituting $P = \frac{\varepsilon V}{c^2}$, then

$$\frac{\varepsilon}{c^2}\frac{dv}{dt} = \frac{e}{c}\vec{V} * \vec{H} \qquad \text{..........1.20}$$

or $$\dot{\upsilon}_x = \omega\upsilon_y, \dot{\upsilon}_y = -\omega\upsilon_x, \dot{\upsilon}_z = 0 \qquad \text{..........1.21}$$

where $\omega = \frac{ecH}{\varepsilon} = \frac{eH}{mc}$ be the angular frequency. Multiplying the 2nd equation of (1.21) by i and adding to the first, we get

$$\frac{d}{dt}\left(\upsilon_x + i\upsilon_y\right) = -i\omega\left(\upsilon_x + i\upsilon_y\right)$$

or $$\upsilon_x + i\upsilon_y = ae^{i\omega t} \qquad \text{..........1.22}$$

where a is a complex constant $= \upsilon_{0t}e^{-i(\omega t+\alpha)}$.

Equation (1.22) can also be written as

$$\upsilon_x + i\upsilon_y = \upsilon_{0t}e^{-i(\omega t+\alpha)} \qquad \text{..........1.23}$$

where υ_{0t} and α are determined by the initial conditions.

The velocity of the particle in XY-plane will be $\upsilon_{0t} = \sqrt{\upsilon_x^2 + \upsilon_y^2}$

On integrating equation (1.23) we get

$$x = x_0 + r\sin\left(\omega t + \alpha\right)$$
$$y = y_0 + r\cos\left(\omega t + \alpha\right) \qquad \text{..........1.24}$$

Where $$r = r = \frac{\upsilon_{ot}}{\omega} = \frac{\upsilon_{ot}\varepsilon}{ecH} = \frac{cP_t}{eH} \qquad \text{..........1.25}$$

be the radius of the curvature of the path of the particle moving in a magnetic field called **Gyro-radius** or **Cyclotron radius**. P_t is the projection momentum on XY-place.[2]

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From third equation of (1.21). we get $\vartheta_z = \vartheta_{0z}$ and

$$Z = z_0 + \upsilon_{oz} t \qquad \text{..........1.26}$$

Since the force acts perpendicular to υ_y, hence it changes its direction, but not magnitude. As the component υ_x, parallel to the field $\vec{H}$ remains constant, so that particle moves in the direction of $\vec{H}$. Therefore the combined effect for the circular and linear horizontal motion is to produce a **helical** motion with the axis of the helix parallel to $\vec{H}$ shown in figure-3[1]. The velocity component perpendicular to the magnetic field creates circular motion whereas the component of the velocity parallel to the field moves the particle along a straight line and resulting motion is helical.

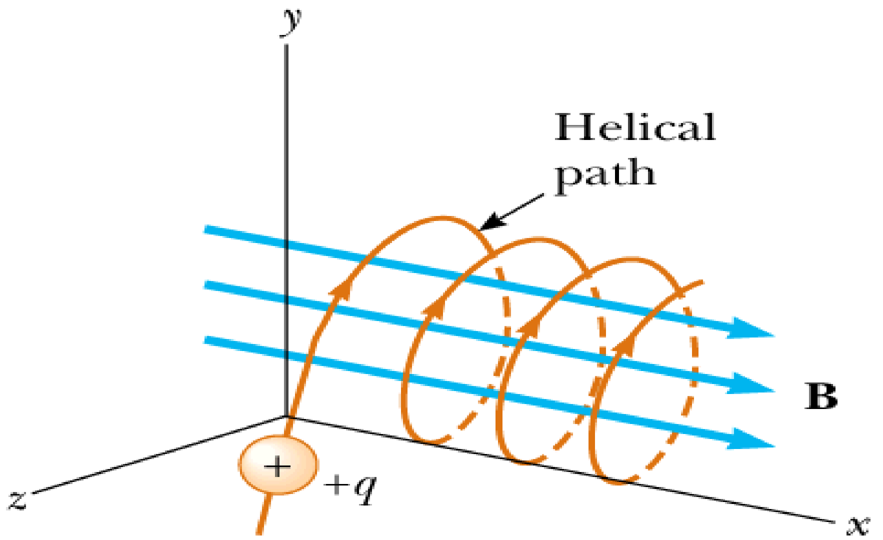
The time required to complete one circle is period of revolution is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi mc}{eH} \qquad \text{..........1.27}$$

The number of revolutions per second by a charged particle in a magnetic field is termed **Gyro Frequency or Cyclotron Frequency** which is given by

$$n = \frac{1}{T} = \frac{eH}{2\pi mc} r.p.s \qquad \text{..........1.28}$$

Equation (1.24) and (1.26) determines that the charge moves in a uniform magnetic field along a **helix** having its axis along the direction of the magnetic field and with a radius r given by equation (1.25). Since velocity of the particle is constant so, the charge has no velocity component along the field i.e $\upsilon_{0z} = 0$ and it moves along a circle in the place perpendicular to the field.[10]



(Fig-3 shows helical path of charged particle)

In a case when the magnetic field varies slowly in magnitude and direction then the motion of charged particle in the plane perpendicular to the magnetic field is periodic and the adiabatic invariant is the integral $I = \frac{1}{2\pi}\oint p_t.dr$, taken over the circumference of a circle.

So,
$$I = \frac{1}{2\pi}\oint p_t.dr + \frac{e}{2\pi c}\oint A.dr \qquad \text{..........1.29}$$

Or
$$I = rP_t + \frac{e}{2c} Hr^2 \qquad \text{..........1.30}$$

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On substituting for r, we get

$$I = \frac{3cP_t^2}{2eH} \qquad \text{..........1.31}$$

This gives for slow variation of H, the tangential momentum P varies proportionally to $\sqrt{H}$

For the motion of charged particle in a quasi uniform field i.e. particle moves in a constant field but not uniform, then the circular orbit is displaced in course of time and the field appears to vary with time while remaining uniform. In this case the momentum transverse to the direction of field varies according to the law $P_t = \sqrt{cH}$, where c is a constant [10].

As the motion in a constant magnetic field, the energy and the square of momentum remains constant so the longitudinal component of momentum will be

$$P_t^2 = P^2 - P_t^2 = P^2 - cH(x, y, z)$$

Since $P_t^2 \geq 0$, then the penetration of the particle into certain portions of space (for $cH > P^2$) may be impossible.[7].

4.3.Example :

For the determination of the Frequency of vibration of a charged oscillator, placed in a constant uniform magnetic field where the proper frequency of vibration of the oscillator is ω_0, (in absence of the field), we solve like this.

The equations of forced vibration of the oscillator in a magnetic field along z-axis are

$$\begin{aligned} \ddot{x} + \omega_0^2 x &= \frac{eH}{mc}\dot{y} \\ \ddot{y} + \omega_0^2 y &= \frac{-eH}{mc}\dot{x} \\ \dot{z} + \omega_0^2 z &= 0 \end{aligned} \qquad \text{..........1.32}$$

Multiplying the second equation by i and combining with the first of equation (1.32), we get

$$\ddot{\zeta} + \omega_0^2 \zeta = -i\frac{eH}{mc}\zeta \quad \text{(Where } \zeta = x + iy\text{)} \qquad \text{..........1.33}$$

The frequency of vibration of the oscillator in a plane perpendicular to the field is

$$\omega = \sqrt{\omega_0^2 + \frac{1}{4}\left(\frac{eH}{mc}\right)} \pm \frac{eH}{2mc} \qquad \text{..........1.34}$$

If the field H is weak, then

$$\omega = \omega_0 \pm \frac{eH}{mc} \qquad \text{.........1.35}$$

The vibration along the direction of the field remains unchanged

4.4.MOTION OF CHARGED PARTICLES IN CONSTANT ELECTRIC AND MAGNETIC FIELD:

Consider the motion of a charged particle 'e' moving in a combination of electric and magnetic fields E and H, both uniform and constant. For this the equation of motion along the direction of H as z – axis will be

$$m\dot{v} = eE + \frac{e}{c}V * H \qquad \text{..........1.36}$$

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$$m\ddot{x} = \frac{e}{c}\dot{y}H$$

or $$m\ddot{y} = eEy\frac{e}{c}\dot{x}H \qquad \text{……….1.37}$$

$$m\ddot{z} = eEz$$

From the third equation, it is noted that the charge moves with uniform acceleration in the z-direction and is given by

$$Z = \frac{eE_z}{2m}t^2 + \upsilon_{0z}t \qquad \text{……….1.38}$$

Multiplying the 2nd equation of (1.37) by i and combining with the first, we get

$$\frac{d}{dt}(\dot{x}+i\dot{y}) + i\omega(\dot{x}+i\dot{y}) = i\frac{e}{m}E_y \qquad \text{……….1.39}$$

or $$\dot{x}+i\dot{y} = \alpha e^{-i\omega t} + \frac{cE_y}{H} \qquad \text{……….1.40}$$

Separating the real and imaginary parts, we get

$$\dot{x} = a\cos\omega t + \frac{cE_y}{H} \qquad \text{……….1.41}$$

$$\dot{y} = -\alpha\sin\omega t$$

The average velocity of the particle along x-axis and y-axis are

$$\overline{\dot{x}} = \frac{cE_y}{H}, \overline{\dot{y}} = 0 \qquad \text{……….1.42}$$

Integrating and choosing the constant of integration so that at t= 0, x = y = 0 and we get

$$x = \frac{\alpha}{\omega}\sin\omega t + \frac{c\varepsilon_y}{H}t$$

$$y = \frac{\alpha}{\omega}(\cos\omega t - 1) \qquad \text{……….1.43}$$

These equations define a **trochoid** which means It is the curve traced out by a point fixed to a circle as it rolls along a straight line shown in figure-4.[7].

Depending on wheather a is large or smaller in absolute value than the quantity $<\dot{z}> \approx \upsilon_0 + \omega_H(x) \approx \frac{\upsilon^2}{\omega_H R}$ the projection of the trajectory on the plane xy.

If $$\alpha = \frac{-c\varepsilon_y}{H}$$,then

$$x = \frac{c\varepsilon_y}{\omega H}(\omega t - \sin\omega t) \qquad \text{……….1.44}$$

$$y = \frac{cE_y}{\omega H}(1 - \cos\omega t) \qquad \text{……….1.45}$$

These gives the projection of the trajectory on the xy plane is a **cycloid** which is denoting a curve trace by a point on a radius of a circle rotating along a straight line or another circle and shown in figure-5.[9].

All the above formulas are valid for the velocity of the particle is small compared with the velocity of light and electric and magnetic fields satisfy

the condition that $\frac{E_y}{H} << 1$

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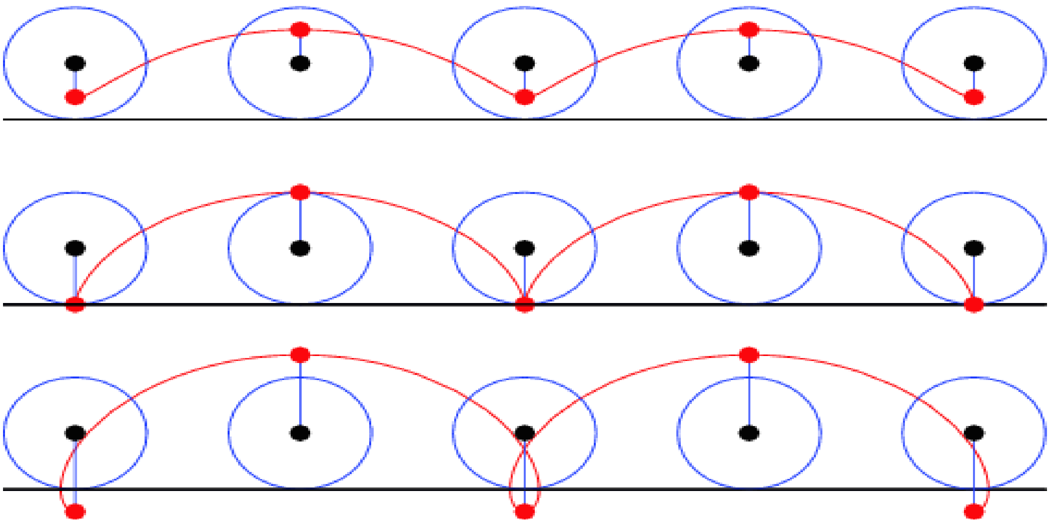
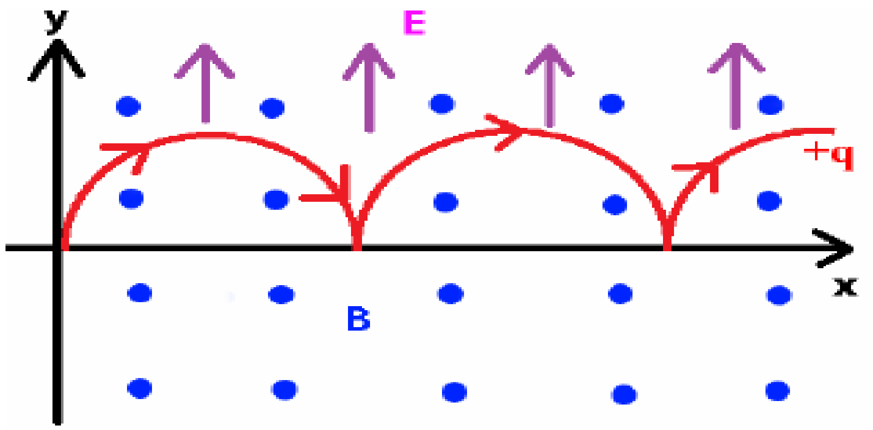


(Fig.-4 shows trochoid path)



(Fig-5 shows cycloid motion is caused by constant magnetic field and constant electric field that are perpendicular)

IV. CONCLUSION

This research deals with fundamental concepts in electromagnetic theory and outlines some basics of numerical modeling. It starts with the relationship between classical electrodynamics and a theory of electromagnetism, List of electromagnetism equations, Maxwell equations, differential equation approach. Some simple electromagnetic theory computational examples are given.The motion of charged particles in various electromagnetic fields results in the working of cathode ray tubes, cathode ray oscilloscope, confinement of plasma, cyclotrons and other accelerators to give a few examples of Classical applications.

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