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An Expression for the Displacement of a Piston with Time in an Expanding Gas

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Abstract: The problem of a piston movement in a confined space filled with gas, say a cylinder, is of interest in many engineering problems. This paper formulates a simple expression for the movement of a piston in a cylinder of expanding gas. A second order differential equation relates the displacement of the piston against time, in terms of the work done (say explosion in the system) and the mass of the piston. This equation has been numerically solved for certain illustrative values.

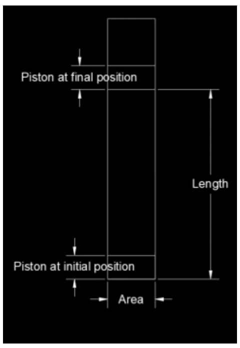
Keywords: Internal Ballistics, Gaussian Error Function, Numerical Solution, Explosion, Boundary Conditions, Third equation of motion

I. INTRODUCTION

This paper formulates an expression for the displacement of a piston in a confined space, a problem of interest in internal ballistics. Design of a more powerful canon can be conceived, if we know when to reinforce an explosion in the barrel. For ensuring that the second reinforcement is correctly timed, we must know, displacement of the cannon ball in a given time after the first explosion. For the purpose of formulation of the problem, we have assumed a piston in place of a cannon ball.

II. FORMULATION

The configuration of the piston is presented in the figure below. The piston of arbitrary cross-section and mass m_p moves in a frictionless cylinder of cross-sectional area A. Below the piston, at its initial position, a small explosion is caused to push the piston up the cylinder for a distance of l.



The work done dw on the piston is given by the relation,

$$dw = pdV \quad \dots (1)$$

where p is the pressure applied on the piston (due to the explosion) and dV is the change of volume of the gas below the piston.

The work done on the piston can also be expressed as,

$$dw = Fdl \quad \dots (2)$$

where F is the force applied on the piston and dl is the incremental displacement of the piston.

On equating the expressions for work done,

$$PdV = Fdl \quad \dots (3)$$

Rearranging we obtain,

$$P = F\,\frac{dl}{dV} \quad \dots (4)$$

Now, F, the force applied can be expressed as the product of the mass of the piston and its acceleration as follows:

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$F = m_p \frac{dv}{dt}$ … (5)

where v is the velocity of the piston.
Substituting this expression for F in equation (4)

$PdV = m_p \frac{dv}{dt} dl$ … (6)

Since dl, is the incremental displacement in time dt,

$dl = vdt$ … (7)

Rewriting,

$PdV = m_p vdv$ … (8)

On integrating both sides,

$\int_{V_i}^{V_f} PdV = m_p \int_0^v vdv$ … (9)

where V_i is initial volume and V_f is the final volume of the gas below the piston.
Hence we obtain,

$P(V_f - V_i) = \frac{m_p v^2}{2}$ … (10)

Or,

$P\Delta V = \frac{m_p v^2}{2}$

Since $P\Delta V$ is the total work done by the system,

$w = \frac{m_p v^2}{2}$ … (11)

Third equation of motion,

$v^2 = u^2 + 2al$ … (12)

Since the piston was initially at rest, initial velocity is 0, and replacing v^2 from eqn. (11), work done may be expressed as:

$w = alm_p$ … (13)

Acceleration of the piston may be expressed as,

$a = \frac{d^2 l}{dt^2}$ … (14)

Substituting expression for a in eqn. (13), work done w is expressed as,

$w = m_p l \frac{d^2 l}{dt^2}$ … (15)

On rearranging,

$l \frac{d^2 l}{dt^2} = \frac{w}{m_p}$ … (16)

The work done by the system is equal to the energy released in explosion or expansion of the gas under the piston. Mass of the piston is known.
This second order differential equation may be solved for the displacement of the piston, l, if initial conditions are known.
Putting,

$\frac{dl}{dt} = \sigma$ … (17)

in the eqn. (16), we get,

$l \frac{d\sigma}{dt} = \frac{w}{m_p}$ … (18)

Substituting dt from eqn. (17),

$l\sigma \frac{d\sigma}{dl} = \frac{w}{m_p}$

And rearranging,

$\sigma d\sigma = \frac{wdl}{lm_p}$ … (19)

We now integrate both sides

$\int \sigma d\sigma = \int \frac{wdl}{lm_p}$

To obtain,

$\frac{\sigma^2}{2} = \frac{w}{m_p} [\ln l + \ln k]$ … (20)

Or,

$\frac{\sigma^2}{2} = \frac{w}{m_p} [\ln kl]$

Where $\ln k$ is a constant.

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On rearranging,

$$\sigma = \sqrt{\frac{2w \ln kl}{m_p}} \quad \dots (21)$$

Now, substituting for σ,

$$\frac{dl}{dt} = \sqrt{\frac{2w \ln kl}{m_p}} \quad \dots (22)$$

On rearranging,

$$\sqrt{\frac{m_p}{2w \ln kl}}\, dl = dt \quad \dots (23)$$

Integrating both sides,

$$\int \sqrt{\frac{m_p}{2w \ln kl}}\, dl = \int dt$$

We obtain,

$$t + c = \int \sqrt{\frac{m_p}{2w \ln kl}}\, dl \quad \dots (24)$$

Where c is a constant, with the following substitutions,

$$\ln kl = f$$

$$e^f = kl$$

$$l = \frac{e^f}{k}$$

$$\mathrm{dl} = \frac{de^f}{k} = \frac{e^f df}{k}$$

Eqn. (24) can be rewritten as,

$$t + c = \frac{1}{k}\sqrt{\frac{m_p}{2w}} \int \frac{e^f df}{\sqrt{f}} \quad \dots (25)$$

We now take the integral,

$$I = \int \frac{e^f df}{\sqrt{f}}$$

Note: solving the given equation in computer, it shows it is a Gaussian erroe function, hence a numerical method must be posed,

And integrate it by parts with u = e^f and v = $\sqrt{f}$

$$\mathrm{I} = \frac{e^f}{\sqrt{f^3}} - [\frac{e^f}{\sqrt{f^5}} - \int \frac{e^f}{\sqrt{f^5}} df]$$

Or,

$$I = \frac{e^f}{\sqrt{f^3}} - \frac{e^f}{\sqrt{f^5}} + [\frac{e^f}{\sqrt{f^5}} - \int \frac{e^f}{\sqrt{f^5}} df] \quad \dots (26)$$

Which can be expressed as an infinite series as follows:

$$I = \lim_{N} \infty \sum_{n=1}^{N} \frac{e^f(-1)^{n+1}}{\sqrt{f^{2n+1}}} \quad \dots (27)$$

Substituting this expression in eqn. (25)

$$t + c = \frac{1}{k}\sqrt{\frac{m_p}{2w}} \lim_{N} \infty \sum_{n=1}^{N} \frac{e^f(-1)^{n+1}}{\sqrt{f^{2n+1}}} \quad \dots (28)$$

Where k and c are constants which can be evaluated based on initial and final states of the system.

Expressing f in terms of y, we obtain the final solution of the differential equation eqn. (16)

$$t + c = \frac{1}{k}\sqrt{\frac{m_p}{2w}} \lim_{N} \infty \sum_{n=1}^{N} \frac{kl(-1)^{n+1}}{\sqrt{(\ln kl)^{2n+1}}} \quad \dots (29)$$

III. CONCLUSION

This paper presents a simple formulation for the displacement of a piston with respect to time a result of explosion or expansion of gas. The second order differential equation that is obtained cannot be solved analytically. An expression for the solution is derived from which numerical solutions can be obtained based on initial conditions. The paper suggests a series summation which can be approximated through computation. The arbitrary constants c and k needed to be solved from boundary conditions.

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