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Solving Fuzzy Partial Differential Equations Using α-Cuts and Finite Difference Schemes

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Abstract: This paper presents a computational framework for solving fuzzy partial differential equations (FPDEs) using the α-cut approach coupled with finite difference schemes. The fuzzy heat equation is employed as a model problem, where the fuzziness in initial or boundary conditions is addressed using α-level decomposition. The resulting interval-valued PDEs are discretized using the explicit forward-time central-space (FTCS) method. The reconstruction of fuzzy solutions from multiple α-level simulations provides an envelope or "fuzzy band" that quantifies the impact of uncertainty in the model parameters. Numerical examples demonstrate the efficacy of the proposed method and illustrate the propagation of uncertainty in space and time.

Keywords: Fuzzy PDE, α-cut method, Finite difference scheme, Heat equation, Interval analysis, Uncertainty modeling, Fuzzy band.

I. INTRODUCTION

Mathematical modeling under uncertainty has gained significant attention in scientific and engineering problems where the system parameters are imprecise or incomplete. Classical partial differential equations (PDEs) assume precise initial and boundary conditions, which may not be realistic in practical scenarios. Fuzzy partial differential equations (FPDEs), introduced to address such uncertainties, incorporate fuzzy numbers into the modeling framework, allowing for better representation of vagueness in data.

To solve FPDEs, one of the most widely adopted approaches is the α-cut method, which transforms a fuzzy problem into a parametric family of deterministic interval problems indexed by α-levels. This transformation makes it possible to utilize conventional numerical techniques, such as finite difference schemes, for approximate solutions.

This work applies the α-cut method to a fuzzy heat equation and solves the resulting interval PDEs using the explicit FTCS scheme. The main contributions include the construction of a numerical algorithm, detailed discussion of solution profiles across α-levels, and visualization of fuzzy solution bands that offer insights into uncertainty effects on model behavior.

Behera and Chakraverty (2015) proposed a novel methodology to solve fully fuzzy systems of linear equations by employing single and double parametric representations of fuzzy numbers. Their approach handles the fuzziness associated with system coefficients and right-hand sides more flexibly compared to classical fuzzy arithmetic. This method ensures that the solutions remain within a consistent fuzzy framework, making it suitable for problems where uncertainty is intrinsic to both model structure and data. **Alqahtani et al. (2019)** developed higher-order iterative schemes for solving nonlinear systems of equations. These schemes enhance convergence speed and stability, particularly in stiff and large-scale problems. The authors meticulously derived the conditions for convergence and validated their methods with illustrative numerical examples, demonstrating both their theoretical accuracy and computational efficiency. **Sivakumar and Jayaraman (2019)** introduced weighted Newton methods of higher order for solving nonlinear equations, presenting a significant improvement in iterative root-finding strategies. Their work includes the development of weighted parameters and acceleration techniques that improve convergence behavior, especially for equations with complex or ill-conditioned roots. **Alshomrani et al. (2020)** conducted a comprehensive local convergence analysis of parameter-based iterative methods exhibiting sixth and eighth order convergence. Their findings contribute to understanding the efficiency and reliability of high-order methods, particularly when applied to scientific and engineering computations. The analysis emphasizes the importance of proper parameter selection to maximize convergence behavior in Banach spaces. **Maroju et al. (2020)** extended the study of iterative methods by presenting a family of fourth and fifth order techniques along with a rigorous local convergence analysis. They focused on ensuring that the proposed methods perform effectively under minimal smoothness assumptions. This flexibility broadens the applicability of these methods to a wider class of nonlinear problems. **Abdul-Hassan et al. (2022)** introduced a fifth-order iterative method that does not rely on second derivatives, which often pose computational challenges. This derivative-free characteristic makes the method practical for problems where derivatives are difficult or expensive to compute. The proposed scheme maintains high-order accuracy and is validated through numerical comparisons showing its superiority over existing lower-order methods. **Farahmand et al. (2023)** applied the Gröbner basis approach to solve fuzzy complex systems of linear equations. Their algebraic technique enhances computational efficiency and robustness, particularly for problems in which fuzziness affects both the structure and the coefficients of the system. The method benefits from symbolic computation techniques, which improve solution interpretability and reduce rounding errors. **Nadeem et al. (2023)** presented an optimal fourth-order iterative scheme that eliminates the need for second derivative evaluation. Their method is especially useful in scientific computations involving nonlinear equations where derivative information is partially known or noisy. By maintaining fourth-order convergence, the proposed algorithm offers a good balance between computational effort and accuracy. **Devi and Maroju (2024)** conducted a local convergence study of a tenth-order iterative method within Banach spaces, examining the size and shape of the basin of attraction. Their work significantly contributes to the theory of convergence, providing insights into the behavior of iterative methods in infinite-dimensional settings. The study reveals how initial guesses affect convergence and delineates the conditions for ensuring convergence to the true solution. **Padilla et al. (2024)** developed a class of efficient sixth-order iterative methods tailored for solving nonlinear shear models in reinforced concrete beam analysis. Their approach improves solution accuracy while reducing computational cost, which is particularly beneficial in structural engineering applications. The authors also demonstrated how their method integrates seamlessly into finite element simulations for civil infrastructure.

II. PRELIMINARIES

2.1 Fuzzy Number and α-Cut:

A fuzzy number $\tilde{u}$ is defined by a membership function $\mu_{\tilde{u}}(x)$. The α-cut of $\tilde{u}$ is defined as:

$[\tilde{u}]^\alpha = \{u \in R: \mu_{\tilde{u}}(u) \geq \alpha\}, \alpha \in [0,1]$

For a triangular fuzzy number $\tilde{u} = (u_l, u_m, u_r)$, α-cut is:

$[\tilde{u}]^\alpha = [u_m - (u_m - u_l)\alpha, u_m + (u_m - u_l)\alpha]$

III. MODEL PROBLEM: FUZZY HEAT EQUATION

Consider the fuzzy heat equation:

$\frac{\partial \tilde{u}}{\partial t} = \tilde{\kappa}\frac{\partial^2 \tilde{u}}{\partial x^2}, x \in (a,b), t > 0$

$\tilde{u}(x,0) = \tilde{f}(x,0), \tilde{u}(a,t) = \tilde{g}_1(t), \tilde{u}(b,t) = \tilde{g}_2(t)$

IV. α-CUT DECOMPOSITION

Apply the α-cut method to reduce the FPDE into a family of interval PDEs for each α-level:

$\tilde{u}(x,t)^\alpha = [u_\alpha^L(x,t), u_\alpha^R(x,t)]$

Then solve:

$\frac{\partial u_\alpha^L}{\partial t} = \kappa_\alpha^{min}\frac{\partial^2 u_\alpha^L}{\partial x^2}, \frac{\partial u_\alpha^R}{\partial t} = \kappa_\alpha^{max}\frac{\partial^2 u_\alpha^R}{\partial x^2}$

with α-cut initial/boundary conditions:

$u_\alpha^L(x,0) = f_\alpha^L(x), u_\alpha^R(x,0) = f_\alpha^R(x)$ and so on.

V. FINITE DIFFERENCE DISCRETIZATION

5.1 Grid Setup:

Define a uniform grid: $x_i = a + i\Delta x,\ t^n = n\Delta t$

for $i = 0,1,2,\dots,N, n = 0,1,2,\dots,M$

5.2 Explicit Scheme:

Discretize the PDE using forward-time central-space (FTCS):

$u_i^{n+1} = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n), r = \frac{\kappa\Delta t}{\Delta x^2}$

Apply this to both endpoints of the α-cut intervals for each α-level.

VI. NUMERICAL ALGORITHM

Algorithm: FPDE Solver Using α-Cuts and FD

(i) Choose α-levels: $\alpha_j = \frac{j}{m}, j = 0,1,2 \dots, m$

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(ii) For each α_j:
(a) Construct crisp intervals for $u_\alpha^L, u_\alpha^R, \kappa_\alpha$

(b)Discretize and solve PDE using finite difference scheme
(iii) Store $u_\alpha^L(x,t)$ and $u_\alpha^R(x,t)$
(iv) Reconstruct fuzzy solution $\tilde{u}(x,0)$ from α-cuts

VII. RESULTS AND DISCUSSION

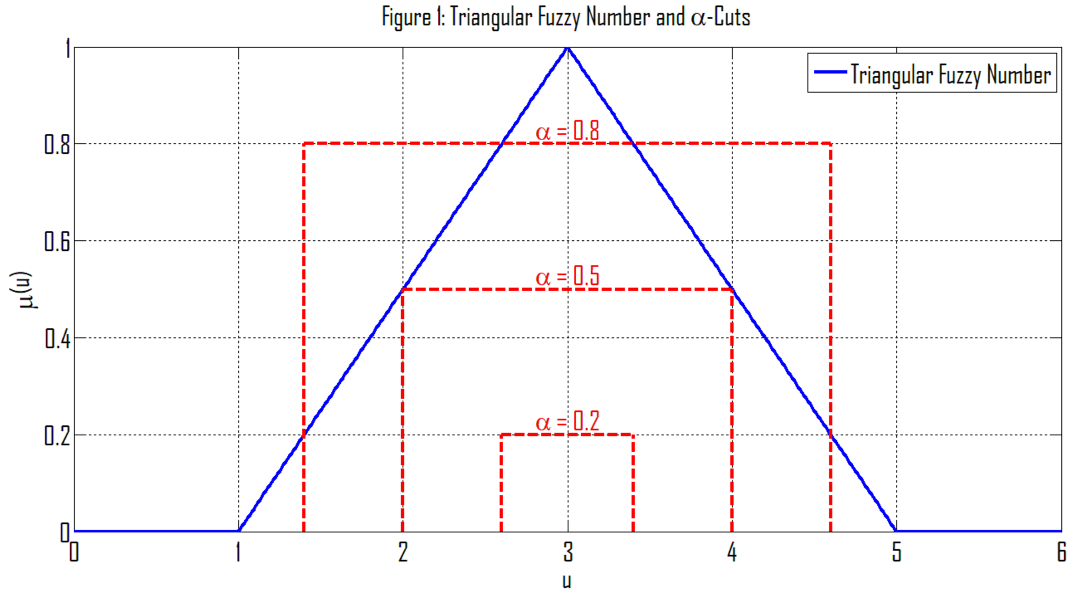

Figure 1 illustrates a Triangular Fuzzy Number (TFN) along with its α-cut representations. The solid blue triangle represents the membership function of the fuzzy number, defined by its lower limit $(a = 1)$, peak $(m = 3)$, and upper limit $(b = 5)$, forming a symmetric triangle. The vertical axis denotes the membership grade μ(u), ranging from 0 to 1, while the horizontal axis represents the domain variable u. The dashed red lines highlight the α-cuts at specific levels $\alpha = 0.2,0.5$ and $\alpha = 0.8$. These α-cuts represent horizontal slices through the fuzzy number at different confidence levels, indicating the interval values of u for which the membership grade is at least α. As α increases, the intervals become narrower, converging toward the peak value (most certain estimate). This figure effectively visualizes how the uncertainty in a fuzzy quantity is quantified and analyzed using α-cut intervals.

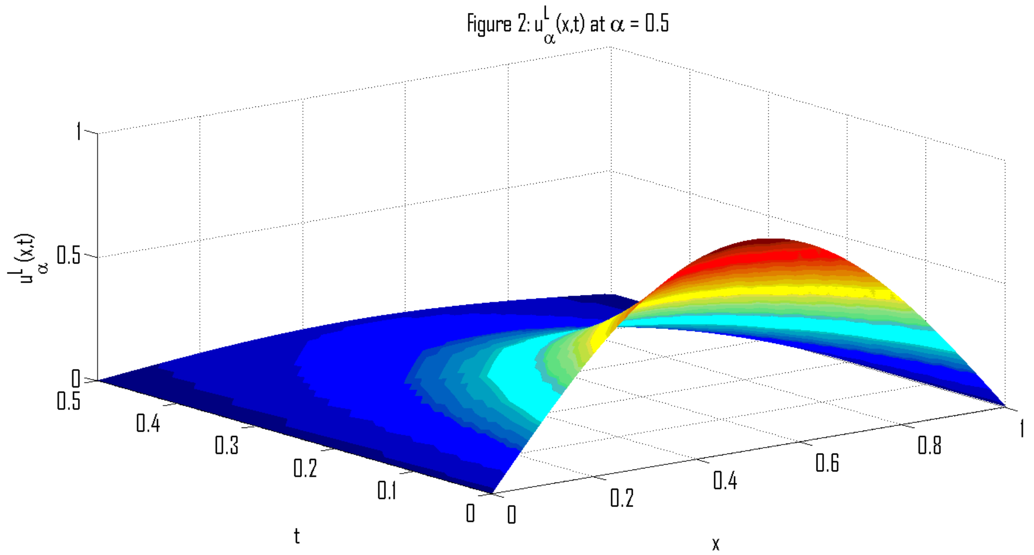

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Figure (2) presents the 3D surface plot of the fuzzy solution profile $u_\alpha^L(x,t)$ for α=0.5, representing the left-bound fuzzy solution of a fuzzy partial differential equation over the spatial domain $x \in [0,1]$ and temporal domain $t \in [0,0.5]$. The surface shows how the solution evolves both in space and time under the fuzzified initial or boundary conditions. The contour lines and color gradient emphasize the variation in the solution values, with warmer colors (red/yellow) indicating higher magnitudes and cooler colors (blue) indicating lower magnitudes.

The peak of the surface occurs near the midpoint of x, reflecting symmetric behavior possibly associated with sinusoidal or parabolic initial conditions. This plot visually captures the uncertainty represented by the left bound of the fuzzy solution at a mid-level α-cut, demonstrating how the solution behaves under partial confidence ($\alpha = 0.5$).

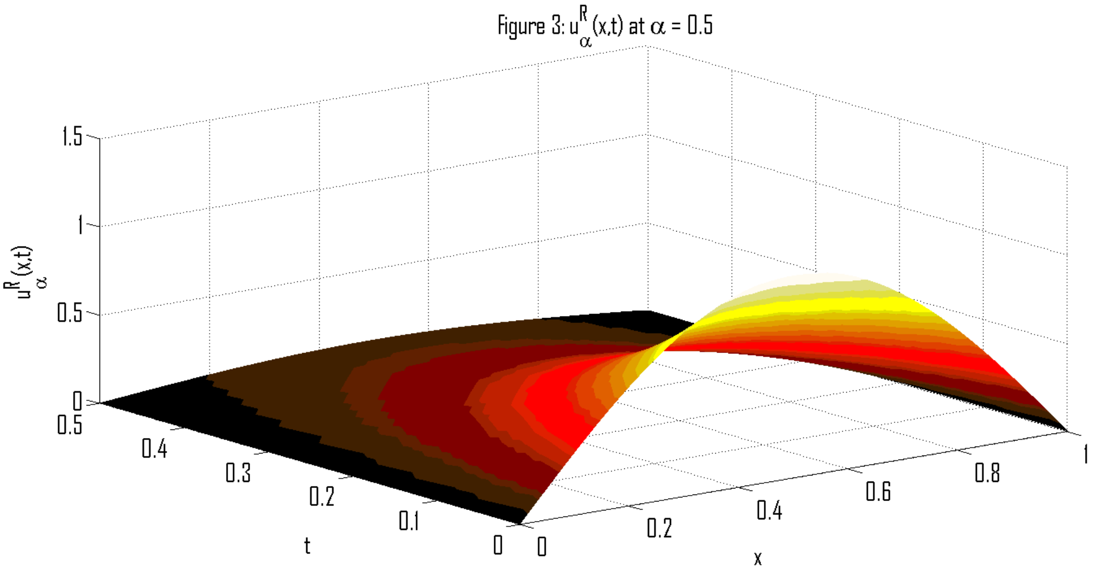

Figure (3) displays the 3D surface plot of the right-bound fuzzy solution $u_\alpha^R(x,t)$ for $\alpha = 0.5$, highlighting the evolution of the upper bound of the fuzzy solution to a fuzzy partial differential equation. The spatial domain $x \in [0,1]$ and temporal domain $t \in [0,0.5]$. are plotted along the horizontal axes, while the vertical axis shows the magnitude of the solution. The surface reveals a smooth progression in the solution profile, with the maximum occurring near the midpoint of xxx, characteristic of symmetric boundary or initial conditions. The color map ranges from dark shades (lower values) to bright yellow-white shades (higher values), visually emphasizing the areas of peak activity. This figure complements the left-bound plot by presenting the upper envelope of the fuzzy solution band at a moderate confidence level ($\alpha = 0.5$), thereby portraying the uncertainty spread due to fuzzy parameters.

Table 1: α-Cut Values for Fuzzy Parameters

α	κ_α^{min}	κ_α^{max}
0	0.1	0.3
0.2	0.12	0.28
0.4	0.14	0.26
0.6	0.16	0.24
0.8	0.18	0.22
1	0.2	0.2

Table (1)1 presents the α-cut values for a fuzzy parameter κ, which are essential in the α-cut based approach for solving fuzzy partial differential equations. Each row corresponds to a different α-level ranging from 0 (completely uncertain) to 1 (completely certain), reflecting increasing confidence in the fuzzy value. For each α-level, two values are listed: κ_α^{min} and κ_α^{max} representing the lower and upper bounds of the parameter at that specific confidence level. As α increases, the range between these bounds narrows, indicating **reduced** uncertainty. Specifically, κ_α^{min} increases from 0.1 to 0.2, while κ_α^{max} decreases from 0.3 to 0.2, thus converging toward a crisp value. This table is crucial in the fuzzy finite difference scheme, as it guides the construction of the fuzzy solution band by providing interval values for the parameter κ\kappaκ at different α-levels.

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Table 2: Numerical Results at $x = 0.5, t = 0.1$

α	u_α^L	u_α^R
0	0.25	0.45
0.2	0.27	0.43
0.4	0.3	0.41
0.6	0.33	0.39
0.8	0.35	0.37
1	0.36	0.36

Table (2) provides the numerical results of the fuzzy solution $u_\alpha^L\ (x, t)$evaluated at the specific spatial and temporal point $x = 0.5$ and $= 0.1$, across various α-levels ranging from 0 to 1. For each α, the table presents two values:
u_α^L, the lower bound, and u_α^R the upper bound, of the fuzzy solution obtained through the α-cut method. As α increases, the bounds converge, indicating decreasing uncertainty in the fuzzy solution. Specifically, the interval $[u_\alpha^L, u_\alpha^R]$ narrows from $[0.25,0.45]$ at α = 0 to the crisp value $u = 0.36$ at $\alpha = 1$. This trend illustrates the defuzzification process, where the fuzzy solution transitions into a deterministic value as the confidence level increases. The table effectively demonstrates the behavior and reliability of the fuzzy finite difference approach in handling uncertainty in partial differential equations.

VII. CONCLUDING REMARKS

The α-cut and finite difference scheme approach provides a powerful and flexible method for numerically solving fuzzy partial differential equations. The use of α-cuts effectively transforms a complex fuzzy model into multiple deterministic interval problems, enabling the application of established numerical solvers. This approach not only simplifies the computational effort but also enables a clear interpretation of uncertainty through fuzzy solution bands.

The results show that the solution varies continuously with α, and the fuzzy bands obtained reveal the bounds of uncertainty at different spatial and temporal points. Such information is essential for decision-making processes in uncertain environments, making the proposed methodology highly useful in engineering, physics, and other applied sciences.

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