IARJSET

ISSN (O) 2393-8021, ISSN (P) 2394-1588

International Advanced Research Journal in Science, Engineering and Technology

Impact Factor 8.066 ⁒ Peer-reviewed / Refereed journal ⁒ Vol. 11, Issue 9, September 2024

DOI: 10.17148/IARJSET.2024.11910

A Glimpse on Homogeneous Ternary Quadratic Diophantine Equation

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Abstract: The focus of this paper is to obtain patterns of integer solutions to homogeneous ternary quadratic diophantine equation given by .The process of obtaining Pythagorean triples from the integer solutions of the considered equation is exhibited.

Keywords: Homogeneous quadratic, Ternary quadratic , Integer solutions, Pythagorean triples , Substitution technique, Factorization method.

I. INTRODUCTION

It is quite obvious that Diophantine equations, one of the areas of number theory, are rich in variety. In particular, the ternary quadratic Diophantine equations in connection with geometrical figures occupy a pivotal role in the orbit of mathematics and have a wealth of historical significance. In this context, one may refer [1-18] for second degree Diophantine equations with three and two unknowns representing different geometrical figures. In this paper, the ternary quadratic Diophantine equation representing cone given by $39(x^2+y^2)+72xy=246z^2$ is studied for determining its integer solutions successfully through elementary algebra.

II. METHOD OF ANALYSIS

The considered homogeneous second degree diophantine equation with three unknowns to be solved is

$$39(x^2+y^2)+72xy=246z^2 \quad (1)$$

By scrutiny, it is observed that (1) is satisfied by the triples of integers $(x,y,z)=(5s,-3s,s),(-5s,3s,s),(29s,-21s,5s),(21s,-25s,5s),(25s,-15s,5s)$. However , there are many more patterns of integer solutions to (1) . The process of determining them is illustrated as follows :

The introduction of the linear transformations

$$x=u+5v,\ y=u-5v,\ z=5P,\ u\neq 5v \quad (2)$$

in (1) leads to the ternary homogeneous quadratic equation

$$u^2+v^2=41P^2 \quad (3)$$

Solving (3) for u,v,P through different ways as shown below and using (2), patterns of integer solutions to (1) are obtained.

Way 1

Assume

$$P=a^2+b^2 \quad (4)$$

Express the integer 41 on the R.H.S. of (3) as

$$41=(4+i5)(4-i5) \quad (5)$$

Substituting (4) & (5) in (3) and employing the method of factorization , consider

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$$u+iv=(4+i5)(a+ib)^2$$

Equating the real and imaginary parts in the above equation ,it is seen that

$$u=4(a^2-b^2)-10ab,$$
$$v=5(a^2-b^2)+8ab.$$

In view of (2) , the corresponding integer solutions to (1) are given by

$$\begin{aligned} x &= 29(a^2-b^2)+30ab, \\ y &= -21(a^2-b^2)-50ab, \\ z &= 5(a^2+b^2). \end{aligned} \qquad (6)$$

From the solutions (6) ,observe that

$$5x+3y=82(a^2-b^2)\ ,$$
$$21x+29y=-820ab.$$

Employing the identity

$$(a^2+b^2)^2-(a^2-b^2)^2=4a^2b^2,$$

notice that

$$\left(\frac{z}{5}\right)^2-\left(\frac{5x+3y}{82}\right)^2=4\left(\frac{21x+29y}{-820}\right)^2$$

$$\Rightarrow (82z)^2-(25x+15y)^2=(21x+29y)^2$$

Thus , the triple $[(21x+29y),(25x+15y),82z]$ represents a Pythagorean triple .

Note 1

Express the integer 41 on the R.H.S. of (3) as

$$41=(5+i4)(5-i4)$$

For this choice , the corresponding integer solutions to (1) are given by

$$x=25(a^2-b^2)+42ab,$$
$$y=-15(a^2-b^2)-58ab,$$
$$z=5(a^2+b^2).$$

Following the procedure as in Way 1, the corresponding Pythagorean triple is represented by

$$[(29x+21y),(15x+25y),82z]$$

Note 2

Let $p,q\,(p>q>0)$ be taken as generators of Pythagorean triangle denoted by PT. Then, the legs and hypotenuse of Pythagorean triangle PT are $p^2-q^2, 2pq$ and p^2+q^2 respectively.

Assume

$$\begin{aligned} f(p,q) &= 5(p^2-q^2)+4(2pq)\ , \\ g(p,q) &= 5(2pq)-4(p^2-q^2), \end{aligned} \qquad (7)$$

where

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$$5^2 + 4^2 = 41.$$

It is worth to mention that

$$41 = \frac{[f(p,q) + i\,g(p,q)][f(p,q) - i\,g(p,q)]}{(p^2+q^2)^2} \tag{8}$$

Write

$$P = (p^2+q^2)^2 (a^2+b^2) \tag{9}$$

Substituting (8) & (9) in (3) and applying factorization , consider

$$u + i\,v = \frac{[f(p,q) + i\,g(p,q)]}{(p^2+q^2)} (p^2+q^2)^2 (a+i\,b)^2$$

On equating the real and imaginary parts in the above equation ,we get

$$u = (p^2+q^2)[(a^2-b^2) f(p,q) - 2\,a\,b\,g(p,q)]\,,$$
$$v = (p^2+q^2)[(2\,ab) f(p,q) + (a^2-b^2)\,g(p,q)].$$

In view of (2) , the corresponding integer solutions to (1) are given by

$$x = (p^2+q^2)[(a^2-b^2+10\,a\,b) f(p,q) + (5(a^2-b^2) - 2\,a\,b)\,g(p,q)]\,,$$
$$y = (p^2+q^2)[(a^2-b^2-10\,a\,b) f(p,q) - (5(a^2-b^2) + 2\,a\,b)\,g(p,q)]\,,$$
$$z = 5(p^2+q^2)^2 (a^2+b^2).$$

Remark : In (7) ,the integers 5 and 4 may be interchanged.

Way 2

Rewrite (3) as

$$41P^2 - u^2 = v^2 * 1 \tag{10}$$

Assume

$$v = 4(41a^2 - b^2) \tag{11}$$

Express the integer 1 on the R.H.S. of (10) as

$$1 = \frac{(\sqrt{41}+5)(\sqrt{41}-5)}{16} \tag{12}$$

Substituting (11) & (12) in (10) and applying factorization ,consider

$$\sqrt{41}\,P + u = \frac{(\sqrt{41}+5)}{4} 4(\sqrt{41}\,a + b)^2$$

from which we get

$$P = 41a^2 + b^2 + 10\,a\,b\,,$$
$$u = 5(41a^2 + b^2) + 82\,a\,b\,.$$

In view of (2) ,the corresponding integer solutions to (1) are found to be

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$$x = 25*41a^2 - 15b^2 + 82ab,$$
$$y = -15*41a^2 + 25b^2 + 82ab,$$
$$z = 5(41a^2 + b^2 + 10ab).$$

Note 3

It is seen that , in addition to (12) , express the integer 1 on the R.H.S. of (10) as

$$1 = \frac{(\sqrt{41}+4)(\sqrt{41}-4)}{25}$$

The repetition of the above process gives a different integer solutions to (1).

Way 3

Write (3) in the form of ratios as

$$\frac{u+5P}{4P+v} = \frac{4P-v}{u-5P} = \frac{\alpha}{\beta}, \beta > 0 \quad (13)$$

Solving the system of double equations (13) by utilizing the method of cross-multiplication ,we have

$$u = 5(\alpha^2 - \beta^2) + 8\alpha\beta,$$
$$v = -4(\alpha^2 - \beta^2) + 10\alpha\beta,$$
$$P = (\alpha^2 + \beta^2).$$

In view of (2) , the corresponding integer solutions to (1) are given by

$$x = -15(\alpha^2 - \beta^2) + 58\alpha\beta,$$
$$y = 25(\alpha^2 - \beta^2) - 42\alpha\beta,$$
$$z = 5(\alpha^2 + \beta^2).$$

The Pythagorean triple generated in this case is given by

$$[(861x + 1189y), (1025x + 615y), 3362z]$$

Note 4

One may also write (3) in the ratio form as

$$\frac{u+4P}{5P+v} = \frac{5P-v}{u-4P} = \frac{\alpha}{\beta}, \beta > 0$$

In this case , the corresponding integer solutions to (1) are found to be

$$x = -21(\alpha^2 - \beta^2) + 50\alpha\beta,$$
$$y = 29(\alpha^2 - \beta^2) - 30\alpha\beta,$$
$$z = 5(\alpha^2 + \beta^2).$$

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III .CONCLUSION

In this paper , the homogeneous second degree equation with three unknowns given by $39(x^2+y^2)+72xy=246z^2$ is studied for obtaining its integer solutions through substitution technique and factorization method. One may search for other forms of quadratic equations with multiple variables to determine their integer solutions.

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