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Abaoub Shkheam decomposition method for a nonlinear fractional Volterra-Fredholm integro-differential equations

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Abstract: The exact solution of a nonlinear fractional Volterra-Fredholm integro-differential equation is found in this paper through the successful application of the Abaoub Shkheam decomposition method. These techniques have a wider range of applications due to their dependability and decreased computational effort.

Additionally, analytical approximations can be used to formally determine the solution's behaviour. Lastly, an example is provided in this study to show the reliability and suitability of the suggested methodologies.

Keywords: Abaoub Shkheam transform, Adomian Decomposition Method, A nonlinear Fractional Volterra Fredholm integro-differential equations.

I. INTRODUCTION

Fractional derivatives with various definitions, such as the Riemann- Liouville fractional integral [1], Caputo fractional derivative [2], and Caputo-Fabrizio fractional derivative [3], have been applied to numerous real-world problems recently by numerous scientists. These researchers have demonstrated the effectiveness of employing such non-integer-order and nonlocal kernels to numerically solve various kinds of integral equations and to characterise the dynamics and properties of these problems; see, for instance, [4–12].

It should be noted that nonlinear Volterra-Fredholm integral equations are used in a wide range of fields, such as neural networks [13], the pulses of sound reflections [14], and mathematical physics such as Lane–Emden-type equations [15], and many more can be found, for example, in [16] and references therein.

In this paper, we consider the nonlinear Caputo fractional Volterra-Fredholm integro-differential equations of the form:.

$$ {}^{c}D^{\alpha}u(x) = g(x) + \int_{0}^{x} k_1(x,t)F_1\big(u(t)\big)dt + \int_{0}^{1} k_2(x,t)F_2\big(u(t)\big)dt \ , \qquad (1) $$

with the initial condition:

$$ u^{(i)}(0) = \delta i \, , i = 0,1,\dots,n-1 \, , \qquad (2) $$

where $n-1 < \alpha \le n$ and $n \in \mathbb{N}, u:[0,1] \to \mathbb{R}$,be the continuous function which has to be determined, $g:[0,1] \to \mathbb{R}$, and $k_i:[0,1] \times [0,1] \to \mathbb{R}$ are continuous functions. $F_i:\mathbb{R} \to \mathbb{R}, i = 1,2$ are nonlinear terms and Lipchitz continuous functions. Here ${}^{c}D^{\alpha}$ stands for the Caputo fractional derivative.

II. PRELIMINARIES

Definition 1. [18]

Let a real function $u(x), x > 0$, which is said to be the space $C_\omega, \omega \in \mathbb{R}$, if the there exists a real number $P > \omega$, such that $u(x) = x^P u_1(x)$, where $u_1(x) \in C\,[0,\infty)$.

Definition 2. (Riemann-Liouville fractional integral) [19]

The Riemann Liouville fractional integral of order $\alpha \ge 0$ of function $u(x) \in C_\omega, \omega \ge -1$ is defined as:

$$ {}_{0}^{RL}I_x^{\alpha}\, u(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^x (x-t)^{\alpha-1}\ u(t)dt, & x>0, \quad \alpha \in \mathbb{R}^+ \\ u(x) & \alpha = 0 \end{cases} \qquad (3) $$

Where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 3. (Caputo fractional derivative)[19]

Let $u(x)$ be a function defined on the interval $[a,b]$, and let $n \in N$ be the smallest integer that the order $\alpha, (n-1 < \alpha \leq n)$. the Caputo fractional derivative of order α of $u(x)$, defined by

$$ {}_{a}^{c}D_x^{\alpha} u(x) = \frac{1}{\Gamma(n-a)} \int_a^x \frac{u^{(n)}(t)}{(x-t)^{\alpha+1-n}}\, dt, \qquad (4) $$

where $u^{(n)}(t)$ is the n-th derivative of $u(t)$ with respect to t .

Consequently, we possess the following properties:

- $.I^{\alpha} I^{\beta} u = I^{\alpha+\beta} u, \quad \alpha, \beta > 0$
- $I^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}, \qquad \alpha > 0,\ \beta > -1, x > 0$
- $I^{\alpha} D^{\alpha} u(x) = u(x) - \sum_{k=0}^{n-1} u^{(k)}(0) \frac{x^k}{k!}, \qquad x > 0,\ n-1 < \alpha \leq n$

Definition 4. (Abaoub Shkheam transform) [17]

The Abaoub Shkheam transform is defined over the set of function

$$ \mathcal{B} = \left\{ f(t) : \exists\, N, k_1, k_2 > 0\,, |f(t)| < N e^{\left(\frac{|t|}{k_j}\right)}, \quad if\ \ t \in (-1)^j \times [0, \infty) \right\} $$

by the following formula

$$ Q\{f(t)\} = \int_0^{\infty} f(vt) e^{\frac{-t}{s}} dt = T\,(v,s) $$

Definition 5. (Definition of convolution) [20]

The convolution of piecewise continuous functions $f, g: \mathbb{R} \to \mathbb{R}$, is the function $(f*g): \mathbb{R} \to \mathbb{R}$ given by:

$$ (u*g)(x) = \int_0^x u(t)\, g(x-t)\, dt $$

Theorem 1.

Let $f(t)$ and $g(t)$ be functions having $\mathcal{L}$ -transform $F(s)$, and $G(s)$ respectively, and having the Q-transform $T_1(v,s)$ and $T_2(v,s)$ respectively. The Q-transform of the convolution of $f(t)$ and $g(t)$ is given by

$$ Q\{f(t) * g(t)\} = vT_1(v,s)T_2(v,s) \qquad (5) $$

Theorem 2.

If $f(x)$ is a continuous function, then the Abaoub Shkheam transform of Riemann Liouville fractional integral for $f(x)$ is given by

$$ Q[{}_{0}^{RL}I_x^{\alpha}\, f(x)] = (ws)^{\alpha}. Q[f(x)] \qquad (6) $$

Proof:

Taking the Abaoub Shkheam transform of both sides for the first part of (3), we get

$$ \begin{aligned} Q[{}_{0}^{RL}I_x^{\alpha}\, f(x)] &= Q\left[\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-t}\ f(t)dt\right] \\ &= \frac{w}{\Gamma(\alpha)} Q[x^{\alpha-t} * f(t)] \\ &= \frac{w}{\Gamma(\alpha)} Q[t^{\alpha-1}]. Q[f(t)] \\ &= \frac{w}{\Gamma(\alpha)} (\alpha-1)!\, w^{\alpha-1}. s^{\alpha}. Q[f(t)] \\ &= \frac{w}{\Gamma(\alpha)}. \Gamma(\alpha). w^{\alpha-1}. s^{\alpha}. Q[f(t)] \\ &= (ws)^{\alpha} Q[f(t)] \end{aligned} $$

$$ Q[J_x^{\alpha} f(x)] = (ws)^{\alpha} Q[f(t)] $$

Theorem 3.

If $f(x)$ is a continuous function, then the Abaoub Shkheam transform of Caputo fractional derivative for $f(x)$ is given by

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$$Q\{{}^{c}D_X^{\alpha}\, f(x)\} = \frac{Q[f(x)]}{(ws)^{\alpha}} - \frac{1}{w}\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{(ws)^{\alpha-1-k}} \tag{7}$$

Proof:

Since

$${}^{c}D_X^{\alpha}\{f(x)\} = I_x^{n-\alpha}.\, f^{(n)}(x), \tag{8}$$

taking the Abaoub Shkheam transform of both sides of (8), we get

$$\begin{aligned} Q\{{}^{c}D_X^{\alpha}\, f(x)\} &= Q\{I_x^{n-\alpha} f^{(n)}(x)\} \\ &= (us)^{n-\alpha} Q[f^{(n)}(x)] \\ &= (us)^{n-\alpha}\left[\frac{Q[f(x)]}{(ws)^{n}} - \frac{1}{w}\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{(ws)^{n-1-k}}\right] \\ &= \frac{Q[f(x)]}{(ws)^{\alpha}} - \frac{1}{u}\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{(ws)^{\alpha-1-k}}. \end{aligned}$$

III. ABAOUB SHKHEAM DECOMPOSITION METHOD (QADM)

On both sides of Eq. (1), we apply the Abaoub Shkheam transform:

$$Q[{}^{c}D^{\alpha}u(x)] = Q[g(x)] + Q\left[\int_0^x k_1(x,t)F_1(u(t))dt + \int_0^1 k_2(x,t)F_2(u(t))dt\right], \tag{9}$$

using theorem 3, we get:

$$\frac{Q[u(x)]}{(us)^{\alpha}} - C = Q[g(x)] + Q\left[\int_0^x k_1(x,t)F_1(u(t))dt + \int_0^1 k_2(x,t)F_2(u(t))dt\right], \tag{10}$$

where $C = \frac{1}{u}\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{(us)^{\alpha-k-1}}$. Consequently, the provided equation is equivalent to

$$Q[u(x)] = (us)^{\alpha}\,.\,C + (us)^{\alpha}[Q[g(x)] + (us)^{\alpha}Q\left[\int_0^x k_1(x,t)F_1(u(t))dt + \int_0^1 k_2(x,t)F_2(u(t))dt\right]. \tag{11}$$

The Adomian method is employed to express the solution $u(x)$, as a series shown below

$$u = \sum_{n=0}^{\infty} u_n\,, \tag{12}$$

and the nonlinear function $F_i, i = 1,2$ is decomposed as:

$$F_1 = \sum_{n=0}^{\alpha} A_n, \quad F_2 = \sum_{n=0}^{\alpha} B_n. \tag{13}$$

where A_n , B_n are the Adomian polynomials given by:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\emptyset^n}(F_1\sum_{i=0}^{n}\emptyset^i u_i)\right]_{\emptyset=0}, \quad B_n = \frac{1}{n!}\left[\frac{d^n}{d\emptyset^n}(F_2\sum_{i=0}^{n}\emptyset^i u_i)\right]_{\emptyset=0}. \tag{14}$$

substituting (12), and (13) into (11) we get:

$$Q\left[\sum_{n=0}^{\infty} u_n\right] = (us)^{\alpha}\,.\,C + (us)^{\alpha}\, Q[g(x)] + (us)^{\alpha}\,.\,Q\left[\int_0^x k_1(x,s)\sum_{n=0}^{\infty} A_n\, ds + \int_0^1 k_2(x,s)\sum_{n=0}^{\infty} B_n\, ds\right]. \tag{15}$$

The iterative procedure that results from comparing both sides of equation (15) is as follows:

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$$Q[u_0] = C(us)^{\alpha} + (us)^{\alpha}\, Q\,[\,g(x)]$$

$$Q[u_1] = (us)^{\alpha}\, Q\left[\int_0^x k_1\,(x,s)A_0\,ds + \int_0^1 k_2\,(x,s)B_0\,ds\right]$$

$$Q[u_2] = (us)^{\alpha}\, Q\left[\int_0^x k_1\,(x,s)A_1\,ds + \int_0^1 k_2\,(x,s)B_1\,ds\right] \quad (16)$$

$$\vdots$$

$$Q[u_{n+1}] = (us)^{\alpha}.\,Q\left[\int_c^x k_1\,(x,s)A_n\,ds + \int_0^1 k_2\,(x,s)B_n\,ds\right].$$

Finally, we take the inverse of Abaoub Shkheam transform of both sides of relations (16), and we obtain the required solution (12).

IV. ILLUSTRATIVE EXAMPLES

In this section, we apply the Abaoub Shkheam decomposition method for solving a nonlinear fractional Volterra Fredholm integro differential equations of the second kind.

Example 3.1: Consider

$$^{c}D^{\frac{3}{4}}[u(t)] + \frac{t^2e^t}{5}u(t) = \frac{6t^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)} + \int_0^t e^t su(s)ds + \int_0^1 (4-s^{-3})u(s)ds, \quad (17)$$

with initial condition

$$u(0) = 0. \quad (18)$$

Applying the Abaoub-Shkheam transform to both sides of (17), we get

$$Q[^{c}D^{\frac{3}{4}}u(t) = Q\left[\left(\frac{-t^2e^t}{5}\right)u(t)\right] + Q\left[\left(\frac{6t^{\frac{9}{4}}}{\Gamma(\frac{13}{4}}\right)\right] + Q\left[\int_0^t e^t su(s)ds + \int_0^1 (4-S^{-3})u(s)ds\right].$$

Using theorem 3, and the initial condition (18), we get:

$$Q[u(t)] = (ws^{\frac{3}{4}})(Q\left[\left(\frac{-t^2e^t}{5}\right)u(t)\right] + Q\left[\frac{6t^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)}\right] + Q\left[\int_0^t e^t su(s)ds + \int_0^1 (4-s^{-3})u(s)ds\right]$$

Substituting (12) and (13) into the above equation we get:

$$Q\left[\sum_{n=0}^{\infty} u_n(t)\right] = (ws)^{\frac{3}{4}}(Q\left[\left(\frac{-t^2e^t}{5}\right)\sum_{n=0}^{\infty} u_n(t)\right] + Q\left[\left(\frac{6t^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)}\right)\right] + Q\left[\int_0^t e^t s\sum_{n=0}^{\infty} A_n ds + \int_0^1 (4-S^{-3})\sum_{n=0}^{\infty} B_n ds\right]$$

By matching both sides of the preceding equation, we obtain

$$Q[u_0(t)] = (ws)^{\frac{3}{4}}\, Q\left[\left(\frac{6t^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)}\right)\right]$$

$$Q[u_1(t)] = ws^{\frac{3}{4}}Q\left[\left(\frac{-t^2e^t}{5}\right)u_0(t)\right] + ws^{\frac{3}{4}}Q\left[\int_0^t e^t S\,A_0 ds + \int_0^1 (4-S^{-3})B_0\,ds\right]$$

$$\vdots$$

$$Q[u_{n+1}(t)] = ws^{\frac{3}{4}}Q\left[\left(\frac{-t^2e^t}{5}\right)u_n(t)\right] + ws^{\frac{3}{4}}Q\left[\int_0^t e^t S\,A_n ds + \int_0^1 (4-S^{-3})B_n\,ds\right]$$

By applying the inverse Abaoub-Shkheam transform to the above equation we get:

$$u_0(t) = t^3$$

$$u_1(t) = Q^{-1}\left(ws^{\frac{3}{4}}Q\left[\left(\frac{-te^t}{5}\right)u_0(t)\right] + ws^{\frac{3}{4}}Q\left[\int_0^t e^t s^4\,ds + \int_0^1 (4-S^{-3})S^3 ds\right]\right) = 0$$

$$\vdots$$

$$u_n(t) = 0, \text{for } n \geq 1.$$

Therefore, the obtained solution

$$u(t) = t^3.$$

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V. CONCLUSION

In this paper, we combined the Abaoub-Shkheam transform with the Adomian Decomposition Method (QADM) to solve a nonlinear fractional Volterra-Fredholm integro-differential equation. Our technique proved to be efficient, decreasing computational complexity while assuring accuracy.

The results demonstrate how effective this approach is at resolving these kinds of problems, particularly those involving Caputo fractional derivatives. Its application to more intricate systems and other kinds of fractional derivatives may be investigated in future research.

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