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Generalization of Some Classes of Integrable Riccati differential Equations

Fatma F.S. Omar [1]

Department of Mathematics, Faculty of Science, Gharyan University, Libya[1]

Abstract: We present a solution method for a general Riccati differential equation by imposing relationships between the coefficients of the Riccati differential equation and explaining them through proofs and examples, we can find the general solution to the various forms of Riccati's equation by integration directly after transforming it into a separable differential equation.

Keywords: Riccati equation; Exact solution; General solution; Integrable differential Equations.

I. INTRODUCTION

The Riccati differential equation is a nonlinear first-order equation. It is expressed by the following equation;

$$y' = q_2(x)y^2 + q_1(x)y + q_0(x), \qquad (1)$$

where $q_2(x)$, $q_1(x)$ and $q_0(x)$ are continuous functions and y is the unknown variable. The Riccati equation is widely used in fields like mathematics, physics, engineering [1], [2], and financial mathematics, with applications in areas such as random processes, optimal control, diffusion problems, network synthesis, and quantum mechanics [3]-[5]. One of its strengths is its ability to connect linear quantum mechanics with other physics domains like thermodynamics and cosmology. The Riccati equation also plays an important role in financial mathematics since most interest-rate models contain time-dependent functions [6]-[11].

Solving the Riccati equation analytically is generally not possible, but numerical methods like the Euler and Runge-Kutta methods are commonly used. In this paper, we will provide algebraic solutions to special forms of the Riccati equation through the following theorems:

Theorem 1.1 The Riccati differential equation

$$y' = q'(x)\left(y^2 + \frac{y}{q(x)} + \frac{a}{q^2(x)}\right), \qquad (2)$$

where $q(x)$ is a continuous function in x and a is a constant, and is a separable differential equation.

Proof. Multiplying (2) by $q(x)^2$, we have an equation

$$q^2(x)y' = q^2(x)q'(x)y^2 + q(x)q'(x)y + aq'(x).$$

Adding $q(x)q'(x)y$ to both sides we obtain

$$q^2(x)y' + q(x)q'(x)y = q^2(x)q'(x)y^2 + q(x)q'(x)y + aq'(x) + q(x)q'(x)y,$$
$$q(x)(q(x)y' + q'(x)y) = q'(x)(q^2(x)y^2 + 2q(x)y + a),$$

Finally, let $u = q(x)y$ we obtain a separable differential equation

$$q(x)u' = q'(x)(u^2 + 2u + a),$$
$$\int \frac{du}{(u^2 + 2u + a)} = \int \frac{q'(x)}{q(x)}dx.$$

Example 1. Solve Riccati differential equation

$$y' = y^2cos(x) + ycot(x) + 15\,cot(x)\sec(x).$$

According to the previous theorem, we find, $a = 15, q(x) = \sin(x), q'(x) = cos(x), u = ysin(x)$,

$$\int \frac{du}{(u^2 + 2u + 15)} = \int \frac{cos\,(x)}{sin\,(x)}dx,$$

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$$\frac{\sqrt{14}tan^{-1}\left(\frac{\sqrt{14}(u\ +\ 1)}{14}\right)}{14} = ln|\sin(x)| + k,$$
$$\frac{\sqrt{14}tan^{-1}\left(\frac{\sqrt{14}(ysi\,n(x)\ +\ 1)}{14}\right)}{14} = ln|\sin(x)| + k.$$

Theorem 1.2. The Riccati equation of the form

$$y' = q'(x)\left(q^{n-2}(x)y^2 + \frac{ay}{q(x)} + \frac{b}{q^n(x)}\right), \qquad (3)$$

where $q(x)$ is a continuous function in x and a, b are constant coefficients $n \in \mathbb{R}, n \neq 2$, and is a separable differential equation.

Proof. Multiplying (3) by $q(x)^n$, we have an equation

$$q^n(x)y' = q^{2n-2}(x)q'(x)y^2 + aq(x)^{n-1}q'(x)y + bq'(x),$$

Adding $(n-1)q^{n-1}(x)q'(x)y$ to both sides of the last equation we will have a new one

$$q^n(x)y' + (n-1)q^{n-1}(x)q'(x)y = q^{2n-2}(x)q'(x)y^2 + aq^{n-1}(x)q'(x)y + bq'(x)$$
$$+(n-1)q^{n-1}(x)q'(x)y,$$
$$q(x)[q^{n-1}(x)y' + (n-1)q^{n-1}(x)q'(x)y] = q'(x)[q^{2n-2}(x)y^2,$$
$$+(a+n-1)q^{n-1}(x)y + b].$$

Finally, usage of the substitution $u = q^{n-1}(x)y$ leads to deal with a separable differential equation

$$q(x)u' = q'(x)(u^2 + (a+n-1)u + b),$$
$$\int \frac{du}{(u^2 + (a+n-1)u + b)} = \int \frac{q'(x)}{q(x)}dx.$$

Example 2. Consider the Riccati differential equation

$$y' = e^{x^2}(2x^2+1)\left(xe^{x^2}y^2 + \frac{2}{xe^{x^2}} + \frac{5}{\left(xe^{x^2}\right)^3}\right).$$

From the Theorem 1.2 we have,

$$\int \frac{du}{(u^2 + (2+3-1)u + 5)} = \int \frac{2x^2e^{x^2} + e^{x^2}}{xe^{x^2}}dx,$$

where $n = 3, a = 2, b = 5,\ q(x) = xe^{x^2}, q'(x) = 2x^2e^{x^2} + e^{x^2}, u = yxe^{x^2}$

$$tan^{-1}(u+2) = ln\left|xe^{x^2}\right| + k,$$
$$tan^{-1}(yxe^{x^2} + 2) = ln\left|xe^{x^2}\right| + k.$$

Theorem 1.3. The Riccati differential equation which has a view of

$$y' = \frac{(q'(x))^2}{q^2(x)}y^2 + \left(\frac{q'(x)}{q(x)} - \frac{q''(x)}{q'(x)}\right)y + a, \qquad (4)$$

where $q(x)$ is a continuous function in x and a is constant coefficient, is a separable differential equation.

Proof. Multiplying (4) by $\frac{q'(x)}{q(x)}$,we have an equation

$$\frac{q'(x)}{q(x)}y' = \frac{(q'(x))^3}{(q(x))^3}y^2 + \frac{q'(x)}{q(x)}\left(\frac{q'(x)}{q(x)} - \frac{q''(x)}{q'(x)}\right)y + \frac{q'(x)}{q(x)}a,$$
$$\frac{q'(x)q(x)y' + q(x)q''(x)y - (q'(x))^2y}{q^2(x)} = \frac{(q'(x))^3}{\left(q(x)\right)^3}y^2 + \frac{q'(x)}{q(x)}a$$
$$\left(\frac{q'(x)y}{q(x)}\right)' = \frac{q'(x)}{q(x)}\left(\frac{(q'(x))^2}{\left(q(x)\right)^2}y^2 + a\right).$$

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Finally, usage of the substitution $u = \frac{q'(x)y}{q(x)}$ leads to deal with a separable differential equation

$$u' = \frac{q'(x)}{q(x)}(u^2 + a),$$
$$\int \frac{du}{(u^2 + a)} = \int \frac{q'(x)}{q(x)} dx.$$

Example 3. Solve Riccati differential equation

$$y' = \frac{(3x^2 + 2)^2}{(x^3 + 2x + 1)^2} y^2 + \frac{3x^4 - 6x + 4}{3x^5 + 8x^3 + 3x^2 + 4x + 2} y + 3.$$

From the Theorem 1.3 we have,

$$\int \frac{du}{(u^2 + 3)} = \int \frac{3x^2 + 2}{x^3 + 2x + 1} dx.$$

where $a = 3, q(x) = x^3 + 2x + 1, q'(x) = 3x^2 + 2, u = \frac{3x^2+2}{x^3+2x+1} y,$

$$\frac{\sqrt{3}}{3} tan^{-1}\left(\frac{\sqrt{3}(3x^2 + 2)}{3\,(x^3 + 2x + 1)} y\right) = ln(x^3 + 2x + 1) + k.$$

Theorem 1.4. If $q_2(x), q_1(x), q_0(x)$ are continuous functions in x, $q_2(x) \neq 0$ and $\frac{q_0(x)}{q_2(x)} e^{-2\int q_1(x)dx} = a$, such that a is a constant, then the function

$$y(x) = k\,\sqrt{a}\, e^{\int q_1(x)dx}.\, tan\left(\sqrt{a} \int q_0(x) e^{\int q_1(x)dx}\, dx\right),$$

is a solution to the Riccati equation (1), k is an integration constant.

Proof.

$$y' = q_2(x)y^2 + q_1(x)y + q_0(x),$$
$$y' e^{-\int q_1(x)dx} = q_2(x)y^2 e^{-\int q_1(x)dx} + q(x)y e^{-\int q_1(x)dx} + q_0(x) e^{-\int q_1(x)dx},$$
$$y' e^{-\int q_1(x)dx} - q_1(x)y e^{-\int q_1(x)dx} = q_2(x)y^2 e^{-\int q_1(x)dx} + q_0(x) e^{-\int q_1(x)dx}$$
$$d\left(y e^{-\int q_1(x)dx}\right) = q_2(x)y^2 e^{-\int q_1(x)dx} + q_0(x) e^{-\int q_1(x)dx},$$

Let $u = y e^{-\int q_1(x)dx}$

$$u' = q_2(x)u^2 e^{\int q_1(x)dx} + q_0(x) e^{-\int q_1(x)dx},$$
$$u' = \left(u^2 + \frac{q_0(x) e^{-\int q_1(x)dx}}{q_2(x) e^{\int q_1(x)dx}}\right) q_2(x) e^{\int q_1(x)dx},$$
$$u' = \left(u^2 + \frac{q_0(x)}{q_2(x)} e^{-2\int q_1(x)dx}\right) q_2(x) e^{\int q_1(x)dx},$$

Since $\frac{q_0(x)}{q_2(x)} e^{-2\int q_1(x)dx} = a$, we have

$$\int \frac{du}{u^2 + a} = \int q_0(x) e^{\int q_1(x)dx}\, dx,$$
$$\frac{1}{\sqrt{a}}\, tan^{-1} \frac{u}{\sqrt{a}} = \int q_0(x) e^{\int q(x)dx}\, dx,$$
$$\frac{u}{\sqrt{a}} = tan\left(\sqrt{a} \int q_0(x) e^{\int q_1(x)dx} dx\right),$$
$$u = \sqrt{a}\, tan\left(\sqrt{a} \int q_0(x) e^{\int q_1(x)dx} dx\right),$$
$$y = \sqrt{a}\, e^{\int q_1(x)dx}.\, tan\left(\sqrt{a} \int q_0(x) e^{\int q_1(x)dx} dx\right).$$

Example 4. Solve Euler-Riccati equation

$$y' = e^{-2x^3} y^2 + (3x^2 + 5)y + 7e^{10x}.$$

From the Theorem 1.4 we have

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$$y = \sqrt{a}\, e^{\int q_1(x)dx}.\, tan\left(\sqrt{a}\int q_0(x) e^{\int q_1(x)dx} dx\right)$$

where

$$q_2(x) = e^{-2x^3}, q_1(x) = 3x^2 + 5, q_0(x) = 7e^{10x}\ ,$$

$$y = \sqrt{7}\, e^{x^3+5x}.\, tan\left(\sqrt{7}\int 7e^{10x} e^{\int (3x^2+5)dx} dx\right),$$

$$y = k\sqrt{7}\, e^{x^3+5x}.\, tan\left(7\sqrt{7}\int e^{x^3+10x+5x}\, dx\right),$$

Lemma 1.1 The Riccati equation of the form

$$y' = q'(x)^2 y^2 - \frac{q(x)''}{q(x)'} y + a, \qquad (5)$$

where $q(x)$, $q(x)'$ are continuous functions in x and a is constant coefficients $n \in \mathbb{R}, n \neq 2$, is a separable differential equation.

Proof. Multiplying (5) by $q(x)'$, we have an equation

$$q(x)'y' = q'(x)^3 y^2 - q''(x)y + aq'(x),$$

$$q(x)'y' + q''(x)y = q'(x)^3 y^2 + aq'(x),$$

$$(q'(x)y)' = q'(x)(q'(x)^2 y^2 + a),$$

Finally, usage of the substitution $u = q'(x)y$ leads to deal with a separable differential equation

$$u' = q'(x)(u^2 + a),$$

$$\int \frac{du}{(u^2 + a)} = \int q'(x)\, dx,$$

$$\tfrac{1}{\sqrt{a}} tan^{-1}(\tfrac{u}{\sqrt{a}}) = q(x) + k,$$

$$\frac{1}{\sqrt{a}} tan^{-1}\left(\frac{q'(x)y}{\sqrt{a}}\right) = q(x) + k.$$

II. CONCLUSION

The work demonstrated a direct connection between the coefficient functions of the Riccati equation and the direct integration of the equation to obtain the general solution after using simple mathematical methods to transform it into a separable equation

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