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A Numerical Approach to the Boundary Layer Flow of Williamson Fluids via Similarity Transformation

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Abstract: This study focuses on the laminar boundary layer flow of Williamson fluids over a wedge, utilizing similarity transformation techniques to reduce the governing nonlinear partial differential equations into ordinary differential equations. Scaling group transformation technique is applied to derive similarity solutions, and numerical computations are carried out using MATLAB's ODE solver to explore the effects of Williamson fluid parameters on flow characteristics. Graphical representations of velocity profiles and their slopes are generated for varying dimensionless parameters, demonstrating significant influence on fluid behavior.

Keywords: Scaling group transformation, Williamson fluid, MATLAB ODE solver, Boundary layer flow.

I. INTRODUCTION

The study of boundary layer flow plays a crucial role in understanding fluid behavior over solid surfaces, particularly when dealing with non-Newtonian fluids such as Williamson fluids. Unlike Newtonian fluids, where viscosity remains constant, Williamson fluids exhibit shear-thinning properties, meaning their apparent viscosity decreases with increasing shear rate. This characteristic makes them highly relevant in various industrial applications, including polymer processing, biomedical engineering, and coating technologies.

Similarity transformations serve as a powerful mathematical tool for simplifying the complex partial differential equations governing fluid motion. By converting these equations into a set of ordinary differential equations, the transformation technique facilitates both analytical and numerical solutions, offering deeper insight into fluid dynamics. The application of similarity transformations in boundary layer studies helps reveal the effects of physical parameters, such as viscosity variation and magnetic field interactions, on flow characteristics.

Various numerical methods are employed to solve the reduced ordinary differential equations resulting from similarity transformations. One such approach is the MATLAB ODE solver, which provides accurate computational results for velocity distribution and temperature variations within the boundary layer. Graphical representations obtained through numerical simulations help visualize the effects of crucial parameters, including the Williamson fluid parameter, Prandtl number, and magnetic field strength.

Philip K. H. Ma and W. H. Hui [1], investigates similarity solutions for the two-dimensional unsteady boundary-layer equations using the Lie group transformation method. The research derives group-invariant similarity solutions for unsteady laminar boundary-layer flows, providing a systematic approach to generating new solutions. A nonlinear superposition method is introduced to extend existing solutions, demonstrating how different classes of similarity solutions relate to the full Navier-Stokes equations. The study also explores flow separation phenomena, contributing to a deeper understanding of unsteady boundary-layer behavior.

Khan N. A., Faqiha Sultan [2], studied the magnetohydrodynamic (MHD) flow of Williamson fluid over an infinite rotating disk, incorporating the effects of Soret, Dufour, and anisotropic slip. The Soret effect refers to mass flux induced by temperature gradients, while the Dufour effect describes energy flux generated by concentration gradients. Anisotropic slip, which accounts for differences in slip lengths in streamwise and spanwise directions, plays a crucial role in modifying the flow behavior. To analyze the system, the researchers transformed the governing nonlinear partial differential equations into ordinary differential equations using the von Kármán similarity transformation. A numerical solution was obtained using MATLAB’s bvp4c routine, allowing for the visualization of velocity, temperature, concentration, and other physical quantities through graphs and tables. This study provides valuable insights into heat and mass transfer characteristics in non-Newtonian fluids, which have applications in engineering and industrial processes.

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Ahmed M. Megahed [3], focuses on the Williamson fluid flow over a nonlinearly stretching sheet, incorporating the effects of viscous dissipation and thermal radiation. These factors play a crucial role in heat transfer processes, particularly in industrial applications such as polymer manufacturing, paper production, and petroleum filtering.

Kumar P., Yadav R. S., Makinde O. D. [4], examined the Williamson fluid flow and heat transfer over a permeable stretching cylinder, incorporating the effects of Joule heating and heat generation/absorption. The study aims to understand how these thermal influences impact fluid behavior, particularly in engineering and industrial applications such as polymer processing, biomedical flows, and cooling systems.

Parmar A. [5], investigates the unsteady convective boundary layer flow of MHD Williamson fluid over an inclined porous stretching sheet, incorporating the effects of non-linear radiation and heat source. The study is particularly relevant to engineering and industrial applications, such as polymer processing, cooling systems, and biomedical fluid dynamics.

Wubshet Ibrahim [6], explores the nonlinear convection flow of Williamson nanofluid past a radially stretching surface. The study investigates the effects of various parameters, including electric fields and buoyancy forces, on the fluid flow and heat transfer characteristics.

Patel M, Timol M. G. [7], focuses on the numerical solutions of boundary layer equations for Williamson fluid past a moving plate. The study explores mathematical modeling and computational techniques to analyze the behavior of non-Newtonian fluids in boundary layer flows.

Khan N. A., Sultan F. [8], investigates the Dufour and Soret effects on magnetohydrodynamic (MHD) flow of Williamson fluid over an infinite rotating disk with anisotropic slip. The study explores how differences in slip lengths in streamwise and spanwise directions influence flow, heat, and mass transfer properties. The governing nonlinear partial differential equations are transformed into ordinary differential equations using the von Kármán similarity transformation, and numerical solutions are obtained using MATLAB's bvp4c routine.

Najeeb Alam Khan and **Hassam Khan [9]**, investigates the boundary layer flows of non-Newtonian Williamson fluid. It was published in Nonlinear Engineering in 2014. The study explores the steady boundary layer flow of Williamson fluid, transforming nonlinear partial differential equations into ordinary differential equations using similarity transformations. The authors employ the homotopy analysis method (HAM) to obtain series solutions for four flow problems: Blasius flow, Sakiadis flow, stretching flow, and stagnation point flow.

Ramesh G.K., Gireesha B.J., and Gorla R. [10], investigates the Sakiadis and Blasius flows of Williamson fluid under a convective boundary condition. It was published in *Nonlinear Engineering* in 2015. The study employs boundary layer approximations and similarity transformations to convert governing partial differential equations into nonlinear ordinary differential equations. These equations are then solved numerically using the fourth and fifth-order Runge-Kutta-Fehlberg method. The results indicate that Blasius flow produces a thicker thermal boundary layer compared to Sakiadis flow.

Timol M.G., Patel M. [11], studied magnetohydrodynamic heat and mass transfer in non-Newtonian power-law fluids plays a crucial role in various industrial and engineering applications, including polymer processing, biomedical engineering, and geophysical fluid dynamics. Unlike Newtonian fluids, whose viscosity remains constant, power-law fluids exhibit shear-dependent viscosity, making their behavior significantly more complex. The boundary layer flow past a semi-infinite flat plate under the influence of a magnetic field introduces additional challenges in fluid modeling and simulation. Similarity analysis provides an effective technique to simplify the governing nonlinear partial differential equations, reducing them to a set of ordinary differential equations for efficient numerical or analytical treatment. The study focuses on determining the effects of key parameters such as the power-law index, Hartmann number, Schmidt number, and Prandtl number on velocity, temperature, and concentration profiles within the boundary layer. The interaction between thermal and mass diffusion, known as Soret and Dufour effects, also plays a vital role in transport phenomena.

Timol M.G. [12], investigated of three-dimensional magnetohydrodynamic (MHD) boundary layer flow in non-Newtonian fluids is crucial for understanding complex fluid behavior in engineering and industrial applications. Unlike Newtonian fluids, non-Newtonian fluids exhibit shear-dependent viscosity, making their boundary layer characteristics more intricate. This research focuses on deriving similarity solutions to simplify the governing equations, transforming nonlinear partial differential equations into ordinary differential equations for efficient analysis. The influence of magnetic fields, viscosity variations, and flow geometry is explored to enhance predictive models for fluid dynamics.

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Morgan A. J. A. [13], explores techniques for simplifying complex systems of partial differential equations by reducing the number of independent variables. This approach is particularly useful in mathematical physics and engineering, where such equations frequently arise in modeling fluid dynamics, heat transfer, and wave propagation. By applying transformation methods, the study provides a systematic framework for reducing multidimensional problems into more manageable forms, facilitating analytical and numerical solutions.

Patel M. and Timol M.G. [14], explores the stress-strain relationship in visco-inelastic non-Newtonian fluids, which differ significantly from classical Newtonian fluids due to their complex rheological behavior. The research categorizes various non-Newtonian fluid models based on their mathematical formulations and investigates how stress and strain velocity interact in these fluids.

Timol M.G. [15], studied on similarity solutions for non-Newtonian fluid flows. His research often focuses on boundary layer theory, employing similarity transformations to simplify complex governing equations. These studies contribute to advancements in fluid mechanics, particularly in understanding the behavior of Williamson fluids under various flow conditions.

Mathematical Problem

The equation of motion for incompressible Williamson non-Newtonian fluid, equation of continuity is identical and applying stream function $u = \psi_y$ and $v = -\psi_x$ can be written by Timol M.G. [15] as,

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} = \frac{1}{\rho}\left(\frac{A}{B + u_y} + \mu_\infty\right)_y \psi_{yy} + UU_x \qquad (1)$$

Where $\left(\frac{A}{B+u_y} + \mu_\infty\right)_y = \tau_{yx}$ is Williamson fluid shearing stress, A, B and μ_∞ are material constants specific to the Williamson fluid model. ψ_{yy} is the rate of the strain of the fluids with applying stream function.

With boundary conditions

$$y = 0, \qquad \psi_y = 0, \qquad \psi_x = 0 \qquad (2)$$

$$y = \infty, \qquad \psi_y = U, \qquad (3)$$

Method of solution for mathematical problem

By applying a symmetry-based transformation that depends on just one parameter, we can reduce the number of variables in the equations. This helps simplify the analysis of the system as

$$\bar{x}^* = L^{\beta_1} x^* \ , \qquad \bar{y}^* = L^{\beta_2} y^*, \qquad \bar{\psi}^* = L^{\beta_3} \psi^*$$
$$\overline{\tau_{yx}}^* = L^{\beta_4} \tau_{yx}{}^* \qquad \bar{U}^* = L^{\beta_5} U^* \qquad (4)$$

Where $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ and L are constants.
For the dependent and independent variables. From equation (4) one obtains

$$\left(\frac{\bar{x}^*}{x^*}\right)^{\frac{1}{\beta_1}} = \left(\frac{\bar{y}^*}{y^*}\right)^{\frac{1}{\beta_2}} = \left(\frac{\bar{\psi}^*}{\psi^*}\right)^{\frac{1}{\beta_3}} = \left(\frac{\overline{\tau_{yx}}^*}{\tau_{yx}{}^*}\right)^{\frac{1}{\beta_4}} = \left(\frac{\bar{U}^*}{U^*}\right)^{\frac{1}{\beta_5}} = L \qquad (5)$$

Applying the linear transformation specified in equation (4) to equations (1), we find that the differential equations remain fully invariant under these transformations. Since the exponent $\beta_1 \neq 0$, we consequently derive the following relationships as

$$\beta_1 = 3\beta_2 = \frac{3}{2}\beta_3 = 3\beta_5 \quad and\ \beta_4 = 0 \qquad (6)$$

By replacing equation (6) within equation (5), we obtain the following outcome are

$$\eta = \frac{y}{x^{\frac{1}{3}}}, \qquad \psi x^{\frac{-2}{3}} = f_1(\eta), \qquad U = f_2(\eta)\ x^{-\frac{1}{3}}, \qquad \tau_{yx} = f_3(\eta) \qquad (7)$$

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Deriving Ordinary Differential Equations

Expressing the similarity variables (7) in relation to equations (1)-(3) leads to the formulation of the following ordinary differential equations.

$$\left(f_1(\eta)\right)'^2 - 2f_1(\eta)f_1''(\eta) - 3f_3(\eta) + \eta f_2(\eta)f_2'(\eta) - f_2^{\,2}(\eta) = 0 \quad (8)$$

We suppose that $f_2(\eta) = 1, therefore\ f_2'(\eta) = 0$ then equation (8) becomes

$$\left(f_1(\eta)\right)'^2 - 2f_1(\eta)f_1''(\eta) - 3f_3(\eta) - 1 = 0 \quad (9)$$

with boundary condition are

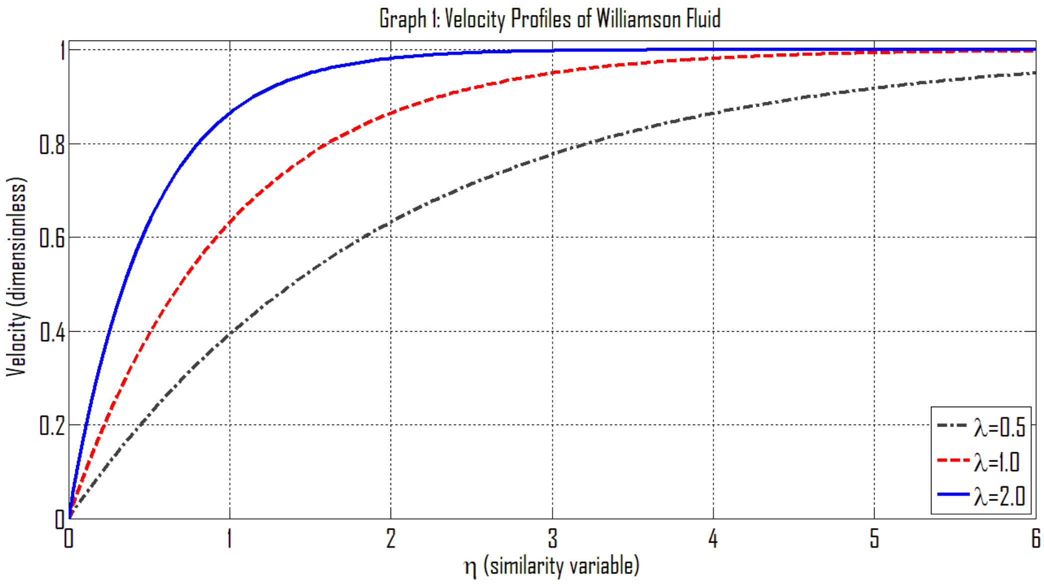
$$For\ \ \eta = 0, \quad f_1'(\eta) = 0, \quad f_1(\eta) = 0 \quad (10)$$

$$For\ \ \eta = \infty, \quad f_1'(\eta) = 1, \quad (11)$$

II. RESULTS AND DISCUSSION

The modified set of nonlinear ordinary differential equations, represented by equations (8) and (9), along with the boundary conditions specified in (10) and (11), is numerically solved using MATLAB's ODE solver. The generated plots illustrate the evolution and distribution of the flow parameters, offering a deeper understanding of the system's dynamic characteristics.

The graph (1) shows the **velocity profiles of Williamson fluid** plotted against the similarity variable η for different values of the Williamson parameter λ. As observed, the velocity increases monotonically with η and approaches the limiting value of 1 (dimensionless free-stream velocity). For larger values of λ (e.g., λ=2.0, blue curve), the fluid velocity rises more sharply and reaches the free-stream value faster, indicating thinner boundary layers. Conversely, for smaller λ (e.g., λ=0.5 , black dash-dot curve), the velocity increases more gradually and takes longer to approach unity, signifying a thicker boundary layer. Thus, the parameter λ strongly influences how quickly the Williamson fluid achieves its free-stream velocity, with higher λ values accelerating the velocity growth.



The graph (2) illustrates the **slope of velocity profiles ($du/d\eta$) of Williamson fluid** as a function of the similarity variable η for different values of the Williamson parameter λ. At η=0, the slope is maximum, showing the steepest velocity gradient near the boundary. As η increases, the slope decreases rapidly and asymptotically approaches zero, indicating that the fluid velocity gradually stabilizes toward the free-stream condition. Higher values of λ (e.g., λ=2.0, blue curve) start with a much sharper slope but decay faster, reflecting a thinner boundary layer and quicker adjustment of velocity. Lower values of λ (e.g., λ=0.5, black dash-dot curve) show smaller initial slopes and slower decay, representing a thicker boundary layer. Thus, increasing λ intensifies the initial velocity gradient near the wall but reduces its influence farther away.

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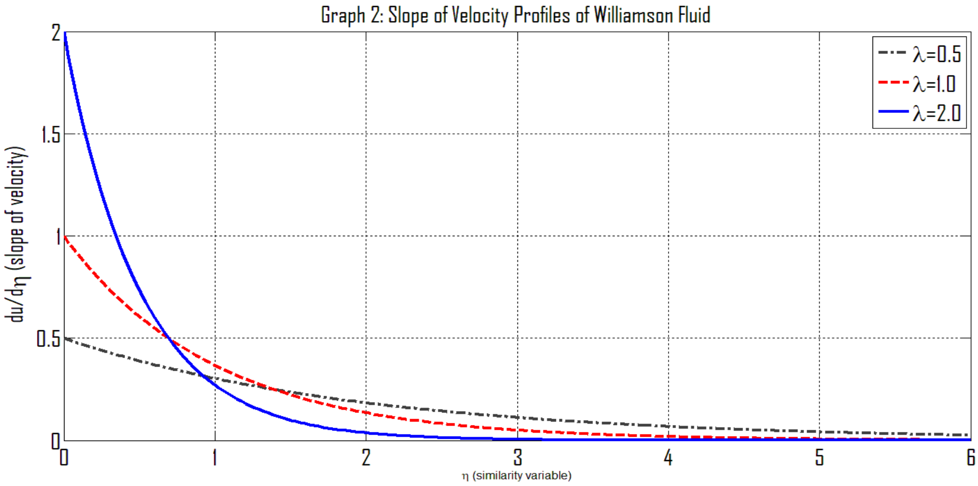

III. CONCLUSION

In this study, the boundary layer flow of Williamson non-Newtonian fluids over a wedge was investigated using similarity transformation techniques. By employing scaling group transformations, the governing nonlinear partial differential equations were successfully reduced to a system of ordinary differential equations, which were then solved numerically using MATLAB's ODE solver.

The analysis revealed that the Williamson fluid parameter significantly influences velocity profiles, shear stress distribution, and boundary layer thickness. Specifically, higher values of the Williamson parameter enhance shear-thinning effects, leading to reduced velocity gradients near the surface and a thinner boundary layer. The graphical results confirmed that variations in thermophysical and flow parameters strongly impact the dynamics of Williamson fluids, aligning with their non-Newtonian shear-dependent nature.

Overall, the study demonstrates the effectiveness of similarity transformations in simplifying complex fluid flow problems and provides a useful framework for predicting the behavior of Williamson fluids in industrial and engineering applications such as polymer processing, coating flows, and biomedical systems.

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