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A STUDY ON ODD EVEN CONGRUENCE LABELLING OF DIGRAPH

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Abstract: This paper investigates the existence of odd-even congruence labelling for various classes of graphs, including a splitting graph of bistars, corona products, tensor products shadow graphs and tadpole graph. A graph $G(V,E)$ is defined as an odd-even congruence graph if there exists a bijective mapping $f: V(G) \to \{1,3,5,\dots\dots,2|V|-1\}$ and an injective mapping $f^*: E(G) \to \{2,4,6,\dots\dots,2|E|\}$ such that for every edge $uv \in E(G)$, the condition $f^*(uv)|\,|f(u)-f(v)$ is satisfied. In addition, constructive proofs and labelling algorithms to demonstrate that these specific graph structures, arising from products and splitting operations, admit such a labelling scheme are provided.

Keywords: Graph labelling, odd-even congruence, splitting graph of bistar graph, corona product, tensor product, tadpole graph and shadow graph.

I. INTRODUCTION

Graph labelling is an assignment of integers to the vertices or edges or both, subject to certain conditions. Since the inception of graceful labelling by Rosa, the field has expanded into various branches, including harmonious, cordial, and prime labelling. These mappings find applications in coding theory, X-ray crystallography, and network addressing.

An odd-even congruence labelling is a specific type of labelling where vertex labels are restricted to odd integers and edge labels to even integers, tied together by a divisibility constraint based on the difference of incident vertex labels. While previous literature has established results for simple paths and cycles, this work focuses on complex graph architectures. Explore the structural properties and congruence feasibility of derived from the corona product of a path and a single vertex. Graphs $Spl(G)$ that introduce shadow vertices to mimic the adjacency of the base graph G. Including corona, tensor, and shadow operations, which often result in a high density of edges requiring unique even divisors. This thesis provides the necessary algebraic constructions to validate these graphs as odd-even congruence graph.

II. BASIC DEFINITION

Definition 2.1:[1]
An *odd even congruence graph* $G=(V,E)$ is defined by a vertex set V where each vertex $v \in V$ is assigned a unique integer label $L(v)$. An edge (u,v) exists in E if and only if$L(u)+L(v) \equiv 1(mod\ 2)$.The edge set is defined a $E = \{\{u,v\} \subseteq V: L(u) \not\equiv L(v)(mod\ 2)\}$.

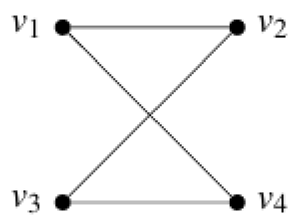

Figure 1: Odd-Even Congruence Graph

Definition 2.2:[10]
The *corona product* $G \odot H$ is obtained by taking one copy of G (with n vertices) and n copies of H and joining the i-th vertex of G to every vertex in the i-th copy of H.

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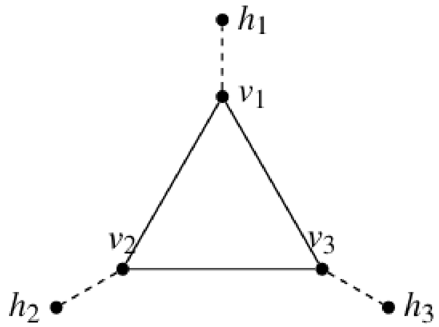

Figure 2: Corona Product

Definition 2.3:[3]
A *bistar graph* $B_{m,n}$ is the graph obtained by joining the central vertices of two stars $K_{1,m}$ and $K_{1,n}$ with an edge.

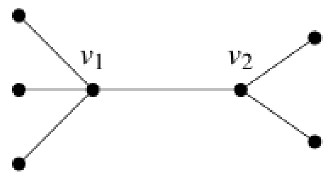

Figure 3: Bistar Graph

Definition 2.4:[1]
The *splitting graph* $S'(G)$ of a graph G is obtained by adding a new vertex v' is adjacent to every neighbor of v in G.

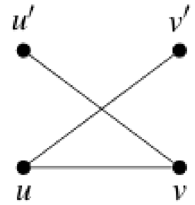

Figure 4: Splitting Graph

Definition 2.5:[1]
The *tensor product* $G \times H$ is a graph where $V(G \times H) = V(G) \times V(H)$ and two vertices (u, v) and(u', v') are if and only if $uu' \in E(G)$ and $vv' \in E(H)$.

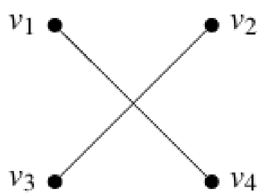

Figure 5: Tensor Product

Definition 2.6:[4]
A *vertex labelling* of a graph G is a function $f: V(G) \to \mathbb{Z}^+$ that assigns labels to the vertices of G.

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Example: 1
Let $V(P_3) = \{v_1, v_2, v_3\}$. Define $f(v_i) = 2i - 1$ for $1 \le i \le 3$. Then the labels are $f(v_1) = 1, f(v_2) = 3, f(v_3) = 5$. The sum of the vertex labels is $\sum f(v_i) = 9$.

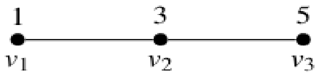

Figure 6: Vertex Labelling of$V(P_3)$

Definition 2.7:[1]
The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G_1 and G_2. For each vertex $v \in V(G_1)$, all neighbors of the corresponding vertex $v' \in V(G_2)$.

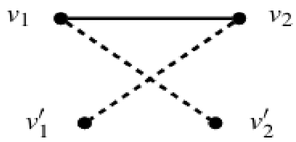

Figure 7: Shadow Graph

Definition 2.8:[4]
An *edge labelling* is a function $g: E(G) \to \mathbb{Z}^+$ that assigns labels to the edges of G, often induced by vertex labels.
Example: 2

Vertex labelling (h) is $h(v_1) = 1, h(v_2) = 3$ and $h(v_3) = 5$.
All vertex labels are distinct odd integers, satisfying the vertex parity constraint.
Edge labelling verification (k) is the edge labels are induced by the absolute difference of the labels of the incident vertices, $k(uv) = |h(u) - h(v)|$, for edge $e_1 = (v_1, v_2)$: $k(e_1) = |1 - 3| = 2$. Since $2 \equiv 0(mod\ 2)$ for edge $e_2 = (v_2, v_3)$: $k(e_2) = |3 - 5| = 2$. Since $2 \equiv 0(mod\ 2)$.

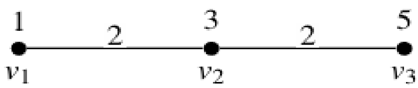

Figure 8: Edge Labelling of P_3

Definition 2.9:[4]
A *congruence labelling* is a labelling scheme where labels are assigned based on modular arithmetic, such as $f(v) \equiv c(mod\ n)$.
Example: 3
Let G be a path graph P_4 with vertices $V = \{v_1, v_2, v_3, v_4\}$. Define the labelling function f such that $f(v_i) \equiv 1(mod\ 2)$. For $1 \le i \le 4$, $f(v_1) = 1, f(v_2) = 3, f(v_3) = 5$ and$f(v_4) = 7$. Each label satisfies the condition as $1,3,5,7 \equiv 1(mod\ 2)$.

Figure 9: Congruence Labelling

Definition 2.10:[5]
An edge$e = (u, v)$ in a graph G is called a *triangle edge* if there exists a vertex $w \in V(G)$ such that $uw \in E(G)$ and $vw \in E(G)$.

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Example: 4

The cycle graph C_3 with vertices $V = \{v_1, v_2, v_3\}$. Let the edge be $e = (v_1, v_2)$. If e is a triangle edge, a third vertex $w = v_3$. Since the edge (v_1, v_3) and (v_2, v_3) both belong to $E(C_3)$, the vertex v_3 completes the triangle. Thus, $e = (v_1, v_2)$ is a triangle edge.

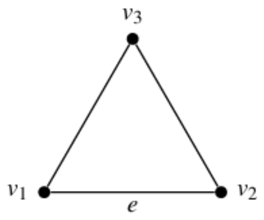

Figure 10: Triangle edge C_3

Definition 2.11:[6]
The *tadpole graph* $T_{r,n}$ is a unicyclic graph obtained one vertex of a cycle C_r with an end-vertex of a path P_n.

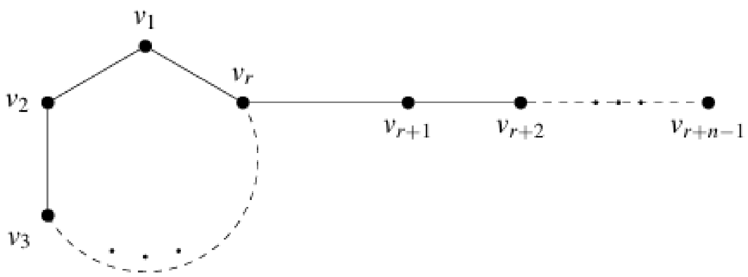

Figure 11: Tadpole Graph

Definition 2.12:[1]
Let $G = (V_G, E_G)$ and $H = (V_H, V_H)$ be two graphs. A function $f: V_G \to V_H$ is called a *bijection* if it satisfies the following two conditions are,

One to one is for any $u, v \in V_G$, if $f(u) = f(v)$, then $u = v$ and onto is for every vertex $w \in V_H$, there exists at least one vertex $v \in V_G$ such that $f(v) = w$. This bijection f must also preserve adjacency, $(u, v) \in E_G \Leftrightarrow \big(f(u), f(v)\big) \in E_H$ for all $u, v \in V_G$.

III. ODD - EVEN CONGRUENCE LABELINGON CORONA PRODUCT, SPLITTING OF BISTAR GRAPH, TENSOR PRODUCT, SHADOW GRAPH AND TADPOLE GRAPH

3.1 Corona Product

Theorem: 3.1.1
The corona product graph $G = P_n \odot C_3$ is an odd even congruence graph for all $n \geq 1$.

Proof

Let $G = P_n \odot C_3$. The graph G is formed by yaking n copies of the cycle C_3 and connecting the i-th vertex of the path P_n to every vertex of the i-th copy of c_3.

The vertex set is defined as $V = \{u_i | 1 \leq i \leq n\} \cup \{v_{i,j} | 1 \leq i \leq n, 1 \leq j \leq 3\}$, where u_i are the vertex of the path P_n and $v_{i,j}$ are the vertices of the i-th copy of C_3.

Number of vertices is $|V| = 4n$, Number of edges is $|E| = (n-1) + 3n + 3n = 7n - 1$ and Maximum edge label is$d = 2|E| = 14n - 2$.

Define the bijection $h: V \to \{1,3,5, \dots \dots, 8n-1\}$ using a scaling factor M, where M is a large odd integer. The labels are assigned as follows

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$$h(u_i) = M(8i-7)$$

$$h(v_{i,1}) = M(8i-5)$$

$$h(v_{i,2}) = M(8i-3)$$

$$h(v_{i,3}) = M(8i-1) \qquad \text{for } 1 \le i \le n.$$

Define the injective mapping $k: E \to \{2,4,6,\ldots\ldots,14n-2\}$. Since the difference between any two adjacent vertex labels $|h(x)-h(y)|$ is a multiple of $2M$, and M is chosen such that it is divisible by all required even labels up to $14n-2$, the congruence condition is satisfied.

For path edges $w_i: ||h(u_i) - h(u_{i+1})|$, cycle edges $c_{i,j}: k(c_{i,j})||h(v_{i,j}) - h(v_{i,k})|$ and joining edges $e_{i,j}: k(e_{i,j})||h(u_i) - h(v_{i,j})|$.

Clearly, for every edge $uv \in E$, the condition $k(uv)$ divides $|h(u) - h(v)|$, which implies $h(u) \equiv h(v)(mod\ k(uv))$.
Hence, the Corona Product graph $P_n \odot C_3$ is an odd-even congruence graph.

Example: 2.1
Let the vertex labels of the corona graph $P_4 \odot C_3$.
Let $G = P_4 \odot C_3$. The vertex labeling h is defined based on the block index $i \in \{1,2,3,4\}$.

Path vertex: $h(u_1) = M(8(1)-7) = 1M$
Cycle vertices: $h(v_{1,1}) = M(8(1)-5) = 3M$
$h(v_{1,2}) = M(8(1)-3) = 5M$
$h(v_{1,3}) = M(8(1)-1) = 7M$

Path vertex: $h(u_2) = M(8(2)-7) = 9M$
Cycle vertices: $h(v_{2,1}) = M(8(2)-5) = 11M$
$h(v_{2,2}) = M(8(2)-3) = 13M$
$h(v_{2,3}) = M(8(2)-1) = 15M$

Path vertex: $h(u_3) = M(8(3)-7) = 17M$
Cycle vertices: $h(v_{3,1}) = M(8(3)-5) = 19M$
$h(v_{3,2}) = M(8(3)-5) = 21M$
$h(v_{3,3}) = M(8(3)-5) = 23M$

Path vertex: $h(u_4) = M(8(4)-7) = 25M$
Cycle vertices: $h(v_{4,1}) = M(8(4)-5) = 27M$
$h(v_{4,2}) = M(8(4)-5) = 29M$
$h(v_{4,3}) = M(8(4)-5) = 31M$

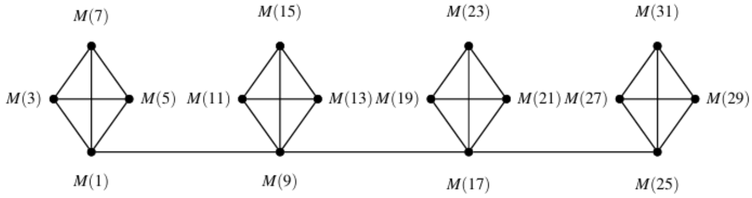

Figure 12: Corona Graph $P_4 \odot C_3$

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3.2 Splitting Graph of a Bistar Graph

Theorem: 3.2.1

The splitting graph of a bistar graph $B_{n,m}$ denoted as $Spl(B_{n,m})$, is an odd-even congruence graph.

Proof

Let $G = Spl(B_{n,m})$. The bistar graph $B_{n,m}$ consists of two central vertices u and v connected by an edge, where u is adjacent to n pendant vertices $\{u_1, u_2, \dots, u_n\}$ and v is adjacent to m pendant vertices $\{v_1, v_2, \dots, v_m\}$.

The splitting graph $\mathrm{Spl}(B_{n,m})$ is obtained by adding a new vertex x' for each vertex $x \in V(B_{n,m})$ such that x' is adjacent to the neighborhood of x.

The vertex set $V(G)$ and edge set $E(G)$ are defined as follows,

$|V| = 2(n + m + 2)$ and $||E| = 3(n + m + 1)$.

Let the vertex set be $V = \{u, v, u', v'\} \cup \{u_i, u_i' | 1 \le i \le n\} \cup \{v_j, v_j' | 1 \le j \le m\}$.

Define the bijection $h: V \to \{1,3,5, \dots, 2p - 1\}$ using a large odd constant L. The labels are assigned as

$h(u) = 1, \quad h(v) = 1 + 2L,$

$$h(u') = 1 + 4L, \quad h(v') = 1 + 6L,$$

$$h(u_i) = 1 + (6 + 2i)L, \quad 1 \le i \le n$$

$$h(u_i') = 1 + (6 + 2n + 2i)L, \quad 1 \le i \le n$$

$$h(v_j) = 1 + (6 + 4n + 2j)L, \quad 1 \le j \le m$$

$$h(v_j') = 1 + (6 + 2n + 2m + 2j)L, \quad 1 \le j \le m$$

Define the injective mapping $k: E \to \{2,4, \dots, 2q\}$. For any edge $xy \in E(G)$,

the difference |h(x) − h(y)| is always a multiple of 2L. Since L is chosen to be sufficiently large and divisible by the required even labels.

For any edge xy, assign a unique even label $k(xy) \in \{2,4, \dots 2q\}$. The condition $k(xy) || h(x) - h(y)|$ is satisfied because the difference is a multiple of 2L. Clearly, $h(x) \equiv h(y)(mod\ k(xy))$ for all $xy \in E(G)$.

Hence, the splitting graph of a bistar graph $Spl(B_{n,m})$ is an odd-even congruence graph.

Example: 2.2

Let the vertex labels of the splitting graph $Spl(B_{2,4})$.

The splitting graph of the bistar $B_{2,4}$. The labeling h follows an odd-even pattern. Vertex labels is odd $u: 1$; $v: 3$; $u_i: \{5,7\}$; $v_j: \{9,11,13,15\}$ and shadow even $u': 2$; $v': 4$; $u_i': \{6,8\}$; $v_j': \{10,12,14,16\}$

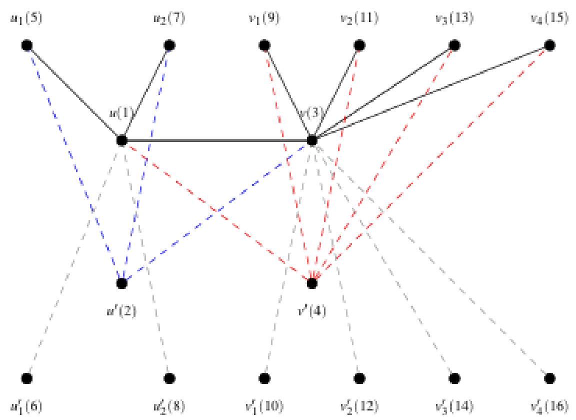

Figure 13: Splitting Graph of a Bistar Graph $Spl(B_{2,4})$.

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3.3 Tensor Product

Theorem: 3.3.1

The tensor product of a star graph and a path graph, $G = K_{1,t} \otimes P_4$, is an odd-even congruence graph for all $t \geq 1$.

Proof

Let $G = K_{1,t} \otimes P_4$.

The vertex set of the star graph $K_{1,t}$ is $\{s_0, s_1, \dots\dots, s_t\}$ where s_0 is the apex vertex. The vertex set of the path P_4 is $\{v_1, v_2, v_3, v_4\}$.

The vertex set $V(G)$ and edge set $E(G)$ are defined as

$$V(G) = \{(s_i, s_j) | 0 \leq i \leq t, 1 \leq j \leq 4\}$$

p =|V|=4(t+1).

$q = |E| = 2|E(K_{1,t})||E(P_4)| = 2(t)(3) = 6t.$

Define the bijection h: V → $\{1, 3, .., 2p - 1\}$ using a scaling factor $L = (2q)!$. For $0 \leq i \leq t$ and $1 \leq j \leq 4$, the labels are assigned as

$h(s_i, v_j) = [4i + (j - 1)] \cdot 2L + 1$

This mapping ensures that all vertex labels are distinct odd integers.

An edge exists between (si, v_j) and (s_i', v_j') if and only if $s_i s_i' \in E(K_{1,t})$ and $v_i v_j' \in E(P_4)$. In this, one vertex must be an image of the apex s_0 and the path indices must be adjacent.

The difference for any edge $e = \{(s_0, v_j), (s_i' v_{j\pm1})\}$ is calculated as

$$|h(s_0, v_j) - h(s_i, v_{j\pm1})| = |[4(0) + (j - 1)] - [4i + (j \pm 1 - 1)]| \cdot 2L$$

$$= |(j - 1) - (4i + j \pm 1 - 1)| \cdot 2L$$

In all cases, the difference is a non-zero multiple of $2L$. Since $L = (2q)!$, any even label $k(e) \in \{2, 4, \dots, 2q\}$ will be a divisor of $2L$.

Thus, assign a unique even label $k(e)$ to each of the $6t$ edges such that $k(e) \mid |h(u) - h(v)|$.

Hence, $G = K_{1,t} \otimes P_4$ is an odd-even congruence graph.

Example: 2.3

Let the vertex labels of the $K_{1,3} \otimes P_4$ is the tensor graph.

$h(s_0, v_j) = 2j - 1$ and $h(s_i, v_j) = 8(i - 1) + 2j$ for $i \in \{1,2,3\}$.

Vertex Labeling h for $K_{1,3} \otimes P_4$

$h(s_0, v_1) = 1$; $h(s_0, v_2) = 3$; $h(s_0, v_3) = 5$; $h(s_0, v_4) = 7$

$h(s_1, v_1) = 2$; $h(s_1, v_2) = 4$; $h(s_1, v_3) = 6$; $h(s_1, v_4) = 8$

$h(s_2, v_1) = 10$; $h(s_2, v_2) = 12$; $h(s_2, v_3) = 14$; $h(s_2, v_4) = 16$

$h(s_3, v_1) = 18$; $h(s_3, v_2) = 20$; $h(s_3, v_3) = 22$; $h(s_3, v_4) = 24$

Edge Labelling k for $K_{1,3} \otimes P_4$

$k((s_0, v_1), (s_1, v_2)) = |1 - 4| = 3$; $k((s_0, v_2), (s_1, v_3)) = |3 - 6| = 3$

$k((s_0, v_2), (s_2, v_1)) = |3 - 10| = 7$; $k((s_0, v_3), (s_2, v_4)) = |5 - 16| = 11$

$k((s_0, v_3), (s_3, v_2)) = |5 - 20| = 15$; $k((s_0, v_4), (s_3, v_3)) = |7 - 22| = 1$

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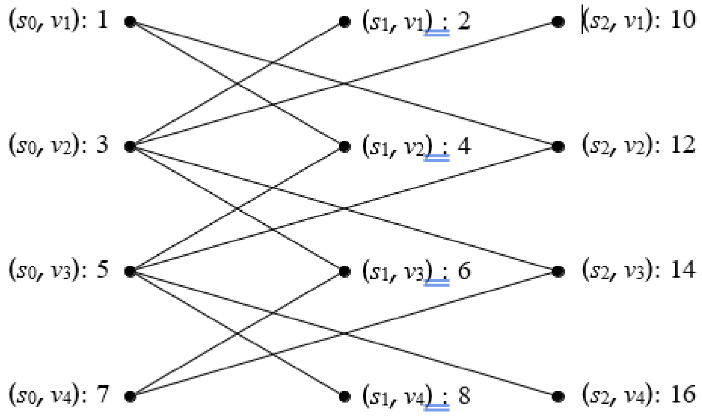

Figure 14: Tensor Graph $K_{1,3} \otimes P_4$.

3.4 Shadow Graph

Theorem: 3.4.1

The shadow graph $D_3(P_t)$ of a path P_t is an odd-even congruence graph for all $t \geq 2$.

Proof

Let $G = D_3(P_t)$. To prove that G is an odd-even congruence graph, let that, a bijective vertex labeling $h : V \rightarrow \{1, 3, \ldots, 2p - 1\}$ and an injective edge labeling $k : E \rightarrow \{2, 4, \ldots, 2q\}$ such that for every edge $uv \in E(G)$, $k(uv)$ divides $|h(u) - h(v)|$.

The shadow graph $D_3(P_t)$ is constructed by taking three copies of the path P_t, say V_1, V_2, and V_3. Let $V_m = \{v_{i,m} \mid 1 \leq i \leq t\}$ for $m = 1,2,3$.

An edge exists between $v_{i,m}$ and $v_{j,r}$ if and only if v_i and v_j are adjacent in the base path P_t. $p = |V| = 3t$, each edge in P_t in $3^2 = 9$ edges in $D_3(P_t)$. Since P_t has $t-1$ edges, $q = |E| = 9(t-1)$.

Define the bijection $h : V \rightarrow \{1, 3, \ldots, 6t - 1\}$ using a scaling factor

$L = (2q)!$. For $1 \leq i \leq t$ and $1 \leq m \leq 3$, the labels are defined as $h(v_{i,m}) = [3(i-1) + (m-1)] \cdot 2L + 1$.

This mapping ensures that each vertex is assigned a unique odd integer.

Consider an arbitrary edge e connecting $v_{i,m}$ and $v_{j,r}$. By the definition of a shadow graph, $|i - j| = 1$. The difference between the vertex labels is

$$|hv_{i,m} - hv_{j,r}| = |[3(i-1) + (m-1)] - [3(j-1) + (r-1)]| \cdot 2L$$
$$= |3(i-j) + (m-r)| \cdot 2L$$

Since $|i-j| = 1$ and $m, r \in \{1, 2, 3\}$, the expression $|3(i-j) + (m-r)|$ is a non-zero integer. Thus, the difference is always a non-zero multiple of $2L$.

Since $L = (2q)!$, any even integer in the set $\{2, 4, \ldots, 2q\}$ is a divisor of $2L$. Therefore, assign a unique even label $k(e) \in \{2, 4, \ldots, 18t - 18\}$ to each of the $9(t-1)$ edges such that $h(u) \equiv h(v) \pmod{k(uv)}$.

Hence, the shadow graph $D_3(P_t)$ is an odd-even congruence graph.

Example: 2.4

Let the vertex labels of the $K_{3,3}$ is shadow graph.

Vertex Labelling is partition V_1 (Odd): $h(v_{1,1}) = 1$, $h(v_{1,2}) = 3$, $h(v_{1,3}) = 5$ and V_2 (Even): $h(v_{2,1}) = 2$, $h(v_{2,2}) = 4$, $h(v_{2,3}) = 6$

Edge Labeling is the edge labels $k(e)$ are induced by the absolute difference of the vertex labels. Since every $u \in V_1$ is adjacent to every $v \in V_2$,

For edge $(v_{1,1}, v_{2,1})$ is $k(e) = |1-2| = 1$, since $1 \equiv 0 \pmod 1$, edge $(v_{1,1}, v_{2,3})$ is $k(e) = |1-6| = 5$, since $5 \equiv 0 \pmod 5$, edge $(v_{1,2}, v_{2,2})$ is $k(e) = |3-4| = 1$,

since $1 \equiv 0 \pmod 1$, edge $(v_{1,3}, v_{2,1})$ is $k(e) = |5-2| = 3$, since $3 \equiv 0 \pmod 3$ and edge $(v_{1,3}, v_{2,2})$ is $k(e) = |5-4| = 1$, since $1 \equiv 0 \pmod 1$.

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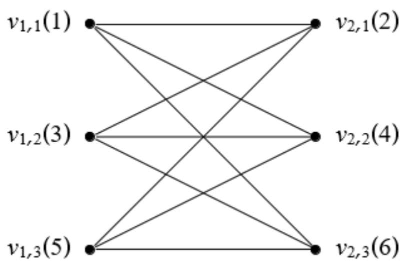

Figure 15: Shadow Graph $D_3(P_2)$.

3.5 Tadpole Graph
Theorem 3.5.1
The Tadpole graph $T_{r,n}$ formed by joining an even cycle C_r to a path P_n at a single vertex, is an odd-even congruence graph for all even $r \geq 4$ and $n \geq 2$.
Proof
Let $G = T_{r,n}$ be the graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_{r+n-1}\}$.
The edge set $E(G)$ consists of the cycle edges, $E_c = \{(v_i, v_{i+1}) : 1 \leq i \leq r-1\} \cup \{(v_i, v_1)\}$ and the path edges, $E_p = \{(v_i, v_{i+1}) : r \leq i \leq r+n-2\}$

Total vertices $|V| = r+n-1$ and edges $|E| = r+n-1$.

Let $d = 2(|V|+|E|) = 4(r+n-1)$.

Define an injective labeling function $h : V(G) \to \{1, 3, 5, \ldots, (2d-1)\}$ as follows For the Cycle Vertices $(1 \leq p \leq r)$

$$h(v_p) = \begin{cases} p & \text{if } p \text{ is odd} \\ d + 7 - p & \text{if } p \text{ is even} \end{cases}$$

For the Path Vertices $(r+1 \leq p \leq r+n-1)$

$$h(v_p) = \begin{cases} d + 9 - p & \text{if } p \text{ is odd} \\ p + 3 & \text{if } p \text{ is even} \end{cases}$$

The edge labels are induced by the function $k(uv) = |h(u) - h(v)|$.
Since d is an even integer and add or subtract odd constants, every $h(v_p)$ results in an odd integer. Furthermore, because d is significantly larger than the number of vertices, the function h is injective.

For any edge $uv \in E(G)$, the induced label $k(uv)$ is the absolute difference between two odd integers, which is always even. By the definition of the difference $h(u) - h(v) \equiv 0 \pmod{k(uv)}$ Which implies $h(u) \equiv h(v) \pmod{k(uv)}$ Thus, the congruence condition is satisfied for all edges.
Therefore, $T_{r,n}$ is an odd-even congruence graph.

Example: 2.5
Tadpole Graph $T_{6,2}$.
Vertex labelling cycle vertices is $h(v_1) = 1, h(v_2) = 69, h(v_3) = 3, h(v_4) = 67, h(v_5) = 5, h(v_6) = 65$ and Path Vertices is $h(v_7) = 66$, $h(v_8) = 11$.
Edge labeling is $k(uv) = |h(u) - h(v)|$.
$k(v_1, v_2) = |1-69| = 68$, $k(v_2, v_3) = |69-3| = 66$, $k(v_3, v_4) = |3-67| = 64$, $k(v_4, v_5) = |67-5| = 62$, $k(v_5, v_6) = |5-65| = 60$, $k(v_6, v_1) = |65-1| = 64$, $k(v_6, v_7) = |65-66| = 1$ and $k(v_7, v_8) = |66-11| = 55$.

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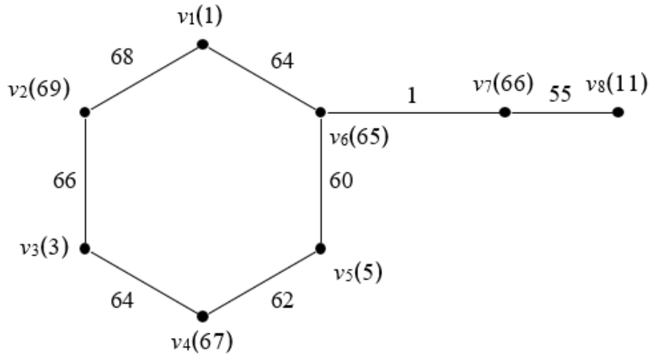

Figure 16: Tadpole Graph $T_{6,2}$

IV. CONCLUSION

In this project, the odd-even congruence labeling for various complex graph families is established. Construct proof and examples for the corona graph, splitting graph on a bistar graph, tadpole graph, tensor graph, and shadow graph are derived. By developing precise mathematical functions using large scaling factors and modular arithmetic, ensured that the unique odd vertex labels and unique even edge labels satisfy the required congruence conditions. These results expand the classification of odd-even congruence graphs and provide a foundation for future research into more intricate graph operations and their modular properties.

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