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ON QUASI PRIME PERIODIC RINGS

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Abstract: A ring R will be called *Quasi-Periodic* if for each x ∈ R, there exist integers k, n, m all depending on x, such that n > m > 0 and $x^n = kx^m$. The concepts of Quasi Periodic Rings are introduced by Howard E.Bell. Now we restrict the condition for k from any integer to a prime number. We define a quasi prime periodic ring as A ring R such that for each x ∈ R, there exist integers n, m and any prime number k all depending on x, such that n > m > 0 and $x^n = kx^m$. Then we compare these quasi prime periodic rings with quasi periodic rings. We give the proof of direct sum of quasi prime periodic rings and nil rings is a quasi prime periodic ring and homomorphic image of a quasi prime periodic ring is a homomorphic ring. Then we study under what conditions quasi prime periodic rings are reduced.

INTRODUCTION

In this introduction we give definition and results used in this paper.

A ring R is said to be *Quasi-Periodic* if for each x ∈ R, there exist integers k, n, m all depending on x, such that n > m > 0 and $x^n = kx^m$. For example $(Z_n, \oplus, \odot)$ and (Z, +, ·) are quasi periodic rings and (Q, +, ·) is not a quasi periodic ring.

The ring *R* is called *reduced* if it has no non-zero nilpotent elements.

A ring R is a *nil ring* if every element of R is nilpotent.

A ring R is *torsion-free* if whenever n is a positive integer and x is an element of R such that nx = 0, then x = 0. For example, Z is a torsion-free ring.

A ring R is *generalised quasi-periodic* if for each x ∈ R there exist distinct positive integers m, n and relatively prime non-zero integers r, s such that $rx^m = sx^n$. It is denoted by G-ring. $(Z_n, \oplus, \odot)$ is a G-ring.

A ring is said to be a *C-ring* if for each x ∈ R, there exists an integer n >1 and an integer k such that $x^n = kx$. For example, $(Z_n, \oplus, \odot)$ is a C-ring.

Results:

Let x be a nilpotent element of a ring R, then 1 + x is a unit in R.

The following results for quasi periodic rings are true for quasi prime periodic rings.

i) Any homomorphic image of a quasi periodic ring is a quasi periodic ring.

ii) Let R be a quasi periodic ring and N be a nil ring. Then R ⊕ N is a quasi periodic ring.

QUASI PRIME PERIODIC RINGS

Definition 1

A ring R will be called *Quasi Prime Periodic* if for each x ∈ R, there exist integers n, m and any prime number k all depending on x, such that n > m > 0 and $x^n = kx^m$.

Examples 2

i) The set of all 2×2 matrices over Z_2 and $(Z_n, \oplus, \odot)$ are quasi prime periodic rings.

ii) (Z, +, ·) is not a quasi prime periodic ring.

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Results 3

i) All the quasi prime periodic rings are quasi periodic rings but the converse need not be true. For (Z, +, ·) is a quasi periodic ring but it is not a quasi prime periodic ring.

ii) All the quasi prime periodic rings are generalised quasi periodic rings but the converse need not be true. For (Z, +, ·) is a generalised quasi periodic ring but it is not a quasi prime periodic ring.

Theorem 4

Direct sum of quasi prime periodic ring and a nil ring is quasi prime periodic ring.

Proof: Let R be a quasi prime periodic ring and N be a nil ring.

To prove R⊕N is a quasi prime periodic ring.

Let $(r, u) \in R \oplus N$ where $r \in R$ and $u \in N$

Now $u \in N$ implies $u^t = 0$ for some $t \in Z^+$.

Let $n > m \geq t$ and $n, m \in Z^+$.

Since $r \in R$, we have $r^n = k\, r^m$ for some prime number k and n and m are positive integers.

Now $(r, u)^n = (r, u) \odot (r,u) \odot (r,u) \odot \cdots \odot (r,u)$ (n times)

$= (r.r.r.r\ldots.r(\text{n times}), u.u.u\ldots.u(\text{n times}))$

$= (r^n, u^n) = (kr^m, 0)$ (Since $u^k = 0$ and $n \geq t$)

$= k\,(r^m, 0) = k\,(r^m, u^m)$ (Since $u^m = 0$)

$= k\,(r.r.r.r\ldots..r(\text{m times}), u.u.u.u\ldots.u(\text{m times}))$

$= k\,(r, u) \odot (r, u) \odot (r, u) \odot \cdots \odot (r, u)$ (m times)

$= k\,(r, u)^m$.

Thus, $(r, u)^n = k\,(r, u)^m$.

Therefore, R ⊕ N is a quasi prime periodic ring.

Theorem 5

If $x \in R$ satisfies $x^n = k\,x^m$ for integers $n > m > 0$ and a prime number k, then for each $j \in Z^+$ and each $s \geq m$ we have $x^{s + j(n-m)} = k^j\, x^s$

Proof:

Given $x \in R$ satisfies $x^n = k\,x^m$ for integers $n > m > 0$ and a prime number k.

We prove this result by induction on j.

When j =1, $x^{s+1(n-m)} = x^{s+n-m} = x^{s-m+n} = x^{s-m}.\, x^n$ (since $s - m \geq 0$)

$= x^{s-m}.\, kx^m = k\,x^{s-m+m} = k\,x^s$

Therefore, the result is true when j = 1.

Now assume that the result is true for $j = t - 1$.

Therefore, $x^{s+(t-1)(n-m)} = k^{t-1}\, x^s$

To prove the result for j = t.

$x^{s+t(n-m)} = x^{s+(t-1)(n-m)+(n-m)} = x^{s+(t-1)(n-m)}.\, x^{n-m}$

$= k^{t-1}\, x^s.\, x^{n-m}$, by induction

$= k^{t-1}\, x^{s+n-m} = k^{t-1}\, x^{s-m}.x^n$, since $s \geq m$

$x^{s+t(n-m)} = k^{t-1}\, x^{s-m}.kx^m = k^{t-1+1}\, x^{s-m+m} = k^t\, x^s$.

Thus, $x^{s+t(n-m)} = k^t\, x^s$. Hence, the result is true for any j.

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Therefore, by induction method, we have $x^{s+j(n-m)} = k^j x^s$ for each $s \geq m$ and $j \in Z^+$

Theorem 6

In any ring R, if x satisfies $x^n = k\,x^m$ for integers $n > m > 0$ and a prime number k then $x^{n-m+1} - kx \in N$. In particular, a quasi periodic ring with no non zero nilpotent elements is a C-ring.

Proof:

Let R be a ring and x satisfies $x^n = k\,x^m$ for integers $n > m > 0$ and a prime number k

We claim that $x^{n-m+1} - kx \in N$.

By theorem 5, for $j = 1 \in Z^+$ and $s = 1 \geq m$, we get $x^{1+(n-m)} = k\,x$.

Now $x^{n-m+1} - kx = kx - kx = 0 \in N$, since 0 is an nilpotent element

Therefore, $x^{n-m+1} - kx \in N$

We claim that quasi periodic ring with no non zero nilpotent elements is a C-ring.

Let R be a quasi periodic ring.

Given 0 is the only nilpotent element.

Therefore, $x^{n-m+1} - kx \in N$

$\Rightarrow x^{n-m+1} - kx = 0 \Rightarrow x^{n-m+1} = kx$

Also $n > m > 0 \implies n - m > 0 \Rightarrow n - m + 1 > 1$

Thus, there exists an integer $n(x) = n - m + 1 > 0$ and $k \in Z$ such that $x^{n(x)} = kx$

Therefore, R is a C-ring.

Theorem 7

Any homomorphic image of a quasi prime periodic ring is a quasi prime periodic ring.

Proof:-

Let R be a quasi prime periodic ring and $f: R \longrightarrow S$ be a onto homomorphism.

Let $y \in S$.

Since $f: R \longrightarrow S$ is onto, there exists an element $x \in R$ such that $f(x) = y$.

Since R is a quasi prime periodic ring, for each $x \in R$ there exists integers n ,m and a prime number k all depending on x such that $n > m > 0$ and $x^n = kx^m$. ------- (i)

Now $y^n = (f(x))^n = (f(x)).\,(f(x)).\,(f(x))\ldots\ldots\,(f(x)\ (n\ \text{times}))$

$= f(x.x.x\ldots..x(n\ \text{times}))$, since f is a homomorphism.

$y^n = f(x^n) = f(kx^m)$ (by (i))

$= k\,f(x^m)$, since f is a homomorphism

$= k\,f(x.x.x\ldots.x(m\ \text{times}))$

$= k\,(f(x)).\,(f(x)).\,(f(x))\ldots\ldots\,(f(x)\ (m\ \text{times}))$

$= k\,(f(x))^m = ky^m$

Since y is an arbitrary element of S, this result is true for all elements in S.

Therefore, S is a quasi prime periodic ring.

Theorem 8

If R is any torsion-free quasi prime periodic ring with 1, then R is reduced.

Proof:

Let R be a torsion-free quasi prime periodic ring with 1.

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To prove that R is reduced, it is enough to prove that R has no non-zero nilpotent elements

Suppose that $u \in R$ with $u^2 = 0 \neq u$.

By the result in the introduction, $1 + u$ is a unit of R.

Then, there exist $m > n > 0$ and $k \in Z^*$ such that $(1 + u)^m = k (1 + u)^n$

Since $1 + u$ is a unit, $(1+u)^{m-n} = k1$.

Take $m – n = g$ then there exist $g \geq 1$ and $k \in Z^*$ such that $(1+u)^g = k1$

Since $u^2 = 0$, $(1+gu) = k1$

Therefore, $gu = (k – 1)1$.

Multiplying by u on both sides, we get $gu^2 = (k – 1) u$.

Since $u^2 = 0$, $(k – 1) u = 0$.

Since $u \neq 0$ and R is the torsion-free ring, $k = 1$.

$\therefore gu = (k – 1)1 = 0$

$\Longrightarrow gu = 0$

Since $u \neq 0$, $g \geq 1$ and R is a torsion-free G-ring, $gu = 0$ is not possible.

Therefore, our assumption is wrong.

That is R has no non zero nilpotent elements.

Hence, R is reduced.

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